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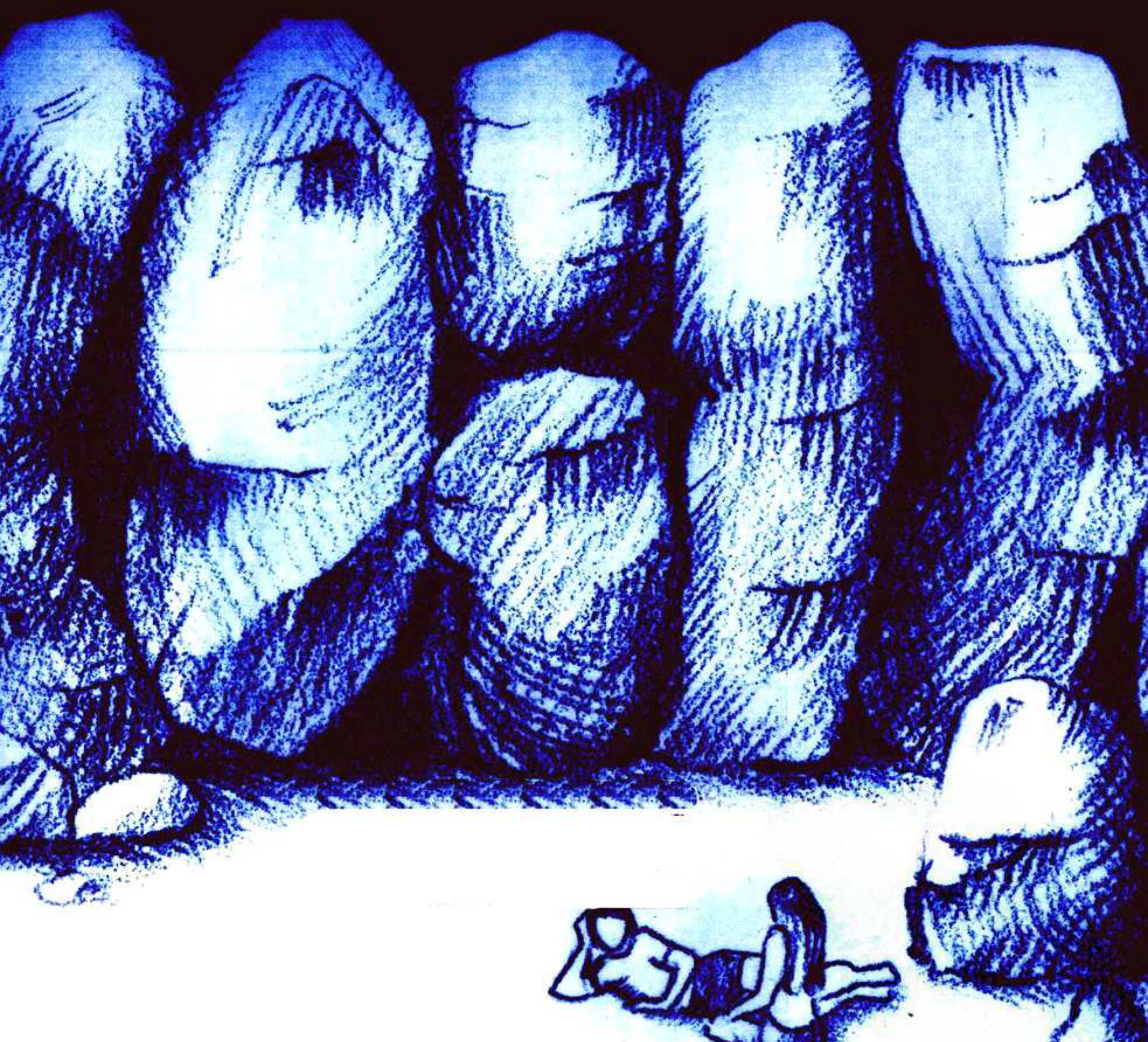
# 研究之美

【美】D. E. KNUTH 著 高博 译



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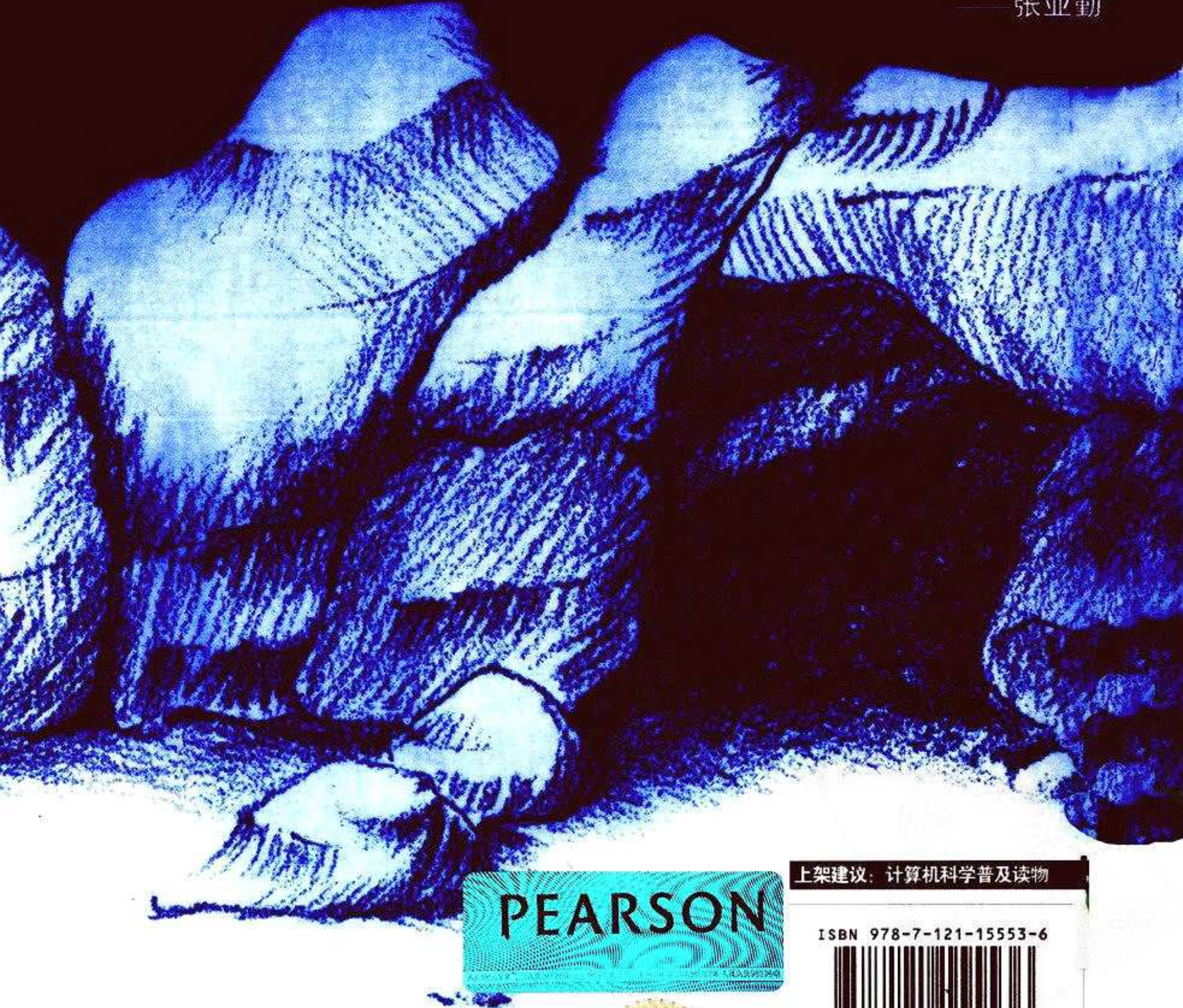
我的主要目标并不是真的要教给读者 Conway 教授的理论，而是想让读者学到，一个人要如何着手来研究出这么一套理论来

——高德纳

我很高兴地看到，高德纳先生在 38 年前写成的一本讨论数学研究的书，能在今天被传译给中国读者。读完后，意料之中的是，本书果然反映了大师的智慧和思路；意料之外的是，原来数学也能以小说的形式来写！

——张亚勤



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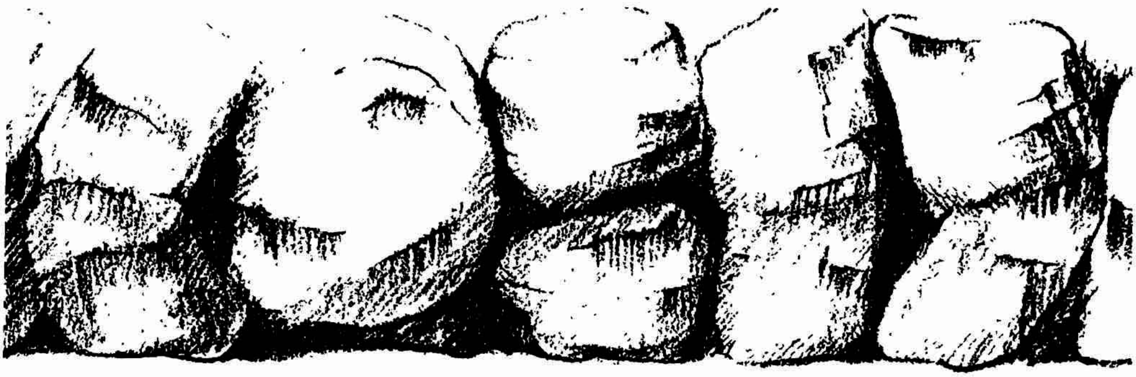
**并获得了终极幸福的故事**



# 研究之美

# SURR NUMBERS

D. E. Knuth 著
高 博 译

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## 内 容 简 介

本书是计算机科学大师、“算法分析之父”高德纳（Donald E. Knuth）在20世纪70年代旅居挪威时撰写的适用于计算机科学的一种全新基础数学结构的情景小品。全书以一对追求自由精神生活的青年男女为主人公，展开了一段对于该种全新结构的发现和构造的对白。在此过程中，本书充分展示了计算机科学的从业人员进行全新领域探索时所必备的怀疑、立论、构造、证明、归纳、演绎等逻辑推理和深入反思的能力。本书可以看作是读懂高德纳的艰深著作《计算机程序设计艺术》和《具体数学》的钥匙。



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# 推荐序

最近几年科学技术和全球市场的变迁，已经越来越清楚地让我们看清了一个事实，那就是持续创新不仅已经成为企业发展的重要基石，甚至已经成为了企业生存的必要前提。这对微软和其他意图在激烈竞争中立于不败之地的企业来说，意味着要拥有马拉松式的创新耐力。

创新的基础是研究，主体是人才。研究机构的建设和创新人才培养，也是我们一直以来的专注和思考。没有研究人才多年如一日地甘于对未知世界的孤独探索，那些能够极大地提高生产力的科技，将不会诞生。

高德纳先生是世界公认的算法大师，他在计算机基础科学方面的很多研究成果，对现在乃至未来的信息产业，已经产生和必然发生深刻影响。他洋洋四卷的《计算机程序设计艺术》，更是被全世界的数学家和工程师奉为圭臬。

我很高兴地看到，他在 38 年前写成的一本讨论数学研究的书，能在今天被传译给中国读者。读完后，意料之中的是，本书果然反映了大师的智慧和思路；意料之外的是，原来数学也能以小说的形式来写！高德纳通过引人入胜的对白和推理来展开情节，为我们展现了数学、方程式、逻辑和算法之美。

研究是一种揭示客观规律的行为，但研究行为自身，也有其独特规律。高德纳先生的这本书，是他几十年智力和思维活动和工业实践的总结，从更高的层次上为我们揭示了研究行为自身的客观规律。从这个意义上说，本书的中文译名《研究之美》，是恰如其分的。

近些年，中国每年毕业的理工科毕业生有七八十万。这些有望成为具备研究能力、掌握研究规律的专门人才，是中国未来竞争力的关键。但遗憾的是，我也看到不少年轻人耐不住枯燥与寂寞，而中途放弃。

为此，衷心祝愿本书能为有志于投身研究事业的读者们，打开一扇趣味之门！

张亚勤
微软全球副总裁
微软亚太研发集团主席
2011 年 12 月 1 日于北京

# PREFACE

Mathematics is the science of patterns, and I especially like the fact that we can use mathematical reasoning to deduce amazing consequences from only two or three simple rules.

One of the most beautiful topics in all of mathematics is the theory of surreal numbers, discovered by John Horton Conway about 1970. A few months after he told me about it, I decided that it would be fun to base a short story around his exciting ideas. Just as an opera consists of wonderful music together with a bit of a plot, I wanted to describe this wonderful mathematics together with a bit of a plot.

I was living in Norway at the time. In January of 1973 I rented a hotel room in downtown Oslo, near where Hendrik Ibsen once lived, in hopes that I might thereby capture some of Ibsen's spirit, and I spent six days writing this little book. On the seventh day I rested. It was the happiest week of my life!

Now, almost forty years later, I'm glad to see that people all over the world have enjoyed this story so much that they have translated it into many different languages. I have been reading many accounts these days about "Doctors Without Borders" and "Engineers Without Borders"; I like to think of myself as a Mathematician Without Borders. For thousands of years mathematics has been a worldwide enterprise, appreciated by people everywhere, and much of this development has occurred in China. Therefore I'm especially happy that this book now appears also in the Chinese language. (Also I'm pleased to note that the translator's name matches my Chinese name 'Gao De Na', which Frances Yao gave to me in 1977 when I was first invited to visit China.)

I sincerely hope that Chinese readers, young and old, will enjoy discovering the beautiful mathematical patterns that Conway has bequeathed to us.

Donald E. Knuth [Gao De Na]

# 序

数学是模式的科学。而我则尤其喜爱这样的事实,就是我们能够运用数学推理,由两三条平凡的规则出发,最终得出令人惊喜的结果。

在所有的数学领域中堪称是最美妙的主题之一,就是超现实数理论。它是由 John Horton Conway 在 1970 年左右发现的。在他告诉我这个理论数月之后,我产生了一个想法:如果能以他的绝妙想法为基础写个短篇故事,那该多么有趣呀。正如歌剧就是美妙的音乐加上那么一点儿情节,我也想在讲述这样美妙的数学时加上那么一点儿情节。

写作此书时我正旅居挪威。1973 年 1 月,我在奥斯陆市区租了一间宾馆的客房,离易卜生[1]的故居很近,所以我指望能通过这种方式沾上点儿易卜生的灵气。然后,我花了六个工作日完成了这本小册子。而到了第七日,我就停下来休息。[2]这是我这辈子最快乐的一星期!

如今,事情已经过去了近四十年。我十分高兴地看到,全世界读者如此地喜爱这个故事,所以他们将它翻译成了很多不同的语言。最近一段时间,我读了不少有关"无国界医生"和"无国界工程师"[3]的故事,我也倾向于认为自己是一名无国界数学家。经历了千百年以后,数学已经成为了一项全球性的事业,吸引着身处所有地域的人为之奋斗,而其中相当一部分的进展都发生在中国。因此,这本书现在出版了简体中文版,是尤其令我欣喜的事。(同时我也愉快地发现译者的名字和我的中文名"高德纳"同姓,而我的中文名字乃是储枫[4]在我 1977 年首次访华时为我起的。)

我衷心希望中国的读者,无论是否仍然年轻,都能够从 Conway 留给我们的美妙数学模式中得到乐趣。

Donald E. Knuth(高德纳)

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[1] 亨德利克·易卜生 (Hendrik Ibsen,1828-1906),挪威戏剧大师,现代现实主义戏剧的创始人,为国人耳熟能详的代表作有《玩偶之家》、《人民公敌》等。——译者注,全书同

[2] 根据《圣经·创世纪》,上帝以六日创生万物,而第七日为安息日。

[3] 无国界组织,指一些专业人士号召摆脱种族、宗教和政治立场,为全人类谋福祉而自发的组织。无国界医生 (Médecins Sans Frontières) 曾获 1999 年诺贝尔和平奖。

[4] 储枫 (Frances Yao),著名学者,1973 年从麻省理工学院取得博士学位。曾在美国多所名校任教,2004 年起在香港城市大学计算机科学系担任系主任。名字中的"Yao"为夫姓姚,其夫为 2000 年图灵奖获奖者姚期智 (Andrew Yao)。

# 译者序

摆在诸位读者面前的，是一本可能会给您的一生带来重大转折契机的书。

这并非故弄玄虚，因为它的作者——计算机算法大师高德纳(Donald E. Knuth)，以三卷本的《计算机程序设计艺术》(The Art of Computer Programming) 一举获得 IEEE 先驱奖和 ACM 图灵奖这两个计算科学界最高奖项。以出版物的成就拿到这两个奖项的，在历史上可谓空前绝后。《计算机程序设计艺术》是无可争辩的神作，毫不夸张地说，全世界和算法分析相关的一切著作，都直接或间接地与这套书有关。但是，能够真正读懂这套书的读者，真是凤毛麟角。且不说里面的数学分析，就算是为讲解算法所使用的程序语言也全是作者自行发明的一种理想机器所使用的，这对于读者的数学功底和抽象思维都提出了非常严苛的要求。所以，有一种说法是，拥有全套《计算机程序设计艺术》的人很多，但是读过十页以上的人则屈指可数。

可是，高德纳本人却一直都认为，再深入的研究也是从最简单的情况入手的。并且一旦掌握了几种做研究的固定套路，即使是普通人也不仅可以着手做研究，甚至还可能做出一流的成果来。这里所说的研究，并非一定是数学研究；而是指从一系列基本的事实或定义出发，通过若干明确的规则，推导出满足这些前提条件的有价值的结论。研究，是最能够体现人类心智的活动，也是创新乃至人类进步的根本源动力。每个人在每天的日常工作和生活中，都在进行着不同程度的研究活动。但是高德纳所主张的研究，既是自觉的、有目的的研究，更是一种工作和生活的基本态度。不仅仅是把研究看作是一种高级的智力活动，更是给心灵带来深层次乐趣的生活方式。唯有如此，才能使研究活动走出象牙塔，打破原本就不存在的条条框框，让每一个人都能领略研究带来的美学享受，提升生活的品质和境界。

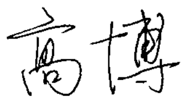
高德纳的博士学位是数学专业，他最得心应手的技术和工具自然也是数学。在本书中，他为了展示研究活动所涉及的方法和思维，也使用了数学作为演示工具。但是，为了尽可能地不让读者陷入具体的技术细节，大师采用了不仅在学术作品中绝无仅有，并且在科普作品中也

绝不多见的情景小品形式。这么一来，从研究问题的提出，到研究体系的构建，再到研究思路的形式，直到研究结论的得出，都通过男女主人公的对白完成。我想，大师这样另辟蹊径的意图，是想让我们把关注的焦点，始终放在研究过程而非具体的数学讨论上。从主人公发现问题的兴奋、遭遇困境的彷徨、探索出路的苦闷、得到结论的欣喜中，我们能够从观众视角充分地体会到研究那点事儿的方方面面，既明确了研究的要素，又了解了研究的方法。更重要的是，从这些对白中我们能够逐渐地明白，对同一问题的研究能够拉近人与人心灵的距离，使得人类之间的欣赏摆脱了低层次的物质追求和利益计算，乃至给生命本身带来升华，真正地使“终极幸福”成为可能。

纯粹从数学知识的角度来看，本书的内容是建立在集合论的基础之上的。数通过集合加以定义，数的顺序映射为集合的关系，而数的运算则映射为集合的运算。定义和规则只有简单的几条，并且非常直观易懂。但是由于数的定义已经从基础上被完全颠覆了，一切有关数的顺序和运算都必须重起炉灶。本书中，男女主人公不断地从基础定义出发，研究出了一系列的中间定理，并总结出若干反复运用的研究套路。最终发现，采用新方法定义的数所组成的集合，竟然是比实数系统更加稠密的数系。并且在这种新的数系中，无穷大量和无穷小量可以像普通的数那样参与运算，而且像“无穷大的一半是多少”这样在传统的实数连续统中没有定义的量，在新的数系中有着很明确的数学意义（实际上，该集合大致相当于现在称为 Grothendieck 宇集的集合，但是不了解这个背景完全不影响阅读）。更妙的是，可以发明一些不符合数的定义，却可以作为中间结果使得一些运算成立或简化的“伪数”，其地位相当于 $\sqrt{-1}$ 这个在人类的认识尚局限于实数的时代曾经一度被排斥的“异端”，可是最终人类却由此出发得出了给数学的进步带来了巨大推动力的复数……总之，内容精彩纷呈，欲知详情，请仔细阅读。大师的手笔，绝不会让你失望！

作为本书的译者，我感觉翻译的过程本身是充满愉悦的享受，可以说是一次与高德纳大师进行心灵对话的宝贵体验。以我个人的体会来说，本书只要仔细阅读，是连高中生都应该可以完全看懂的。即使一时静不下心来去读懂具体的推导过程，看看大师在主人公的对白中有意

插入和强调的那些研究所带来的普世价值，也是大有裨益的。本书成稿的过程中，得到了博文视点符隆美编辑的大力支持，没有她的关照和鼓励，我不可能这么快地完成全书的翻译。承美国南密西西比大学数学系丁玖教授抽出宝贵时间阅读了全书译稿，并提出了数十条专业意见。北京师范大学数学系的赵钊研究员也费心费力，对译稿的可读性和技术问题作了上百处修订。在此，向他们两位表示衷心的谢意。本书成稿过程中，上海交通大学计算科学与技术系的张尧弼教授和窦延平教授、上海交通大学软件学院的陈平教授、SAP 中国的范德成工程师、盛大创新院的刘海平研究员、微软亚洲工程院的王楠工程师、谷歌中国的龚理工程师、美光半导体的赵海源技术顾问、上海申通地铁集团的蒋振伟项目经理等都提出过若干稿件修正意见，在此一并致谢。当然，由于本人能力所限，本书的缺点和不足仍在所难免，这些理应由我一人负责。为向高德纳大师致敬，本书简体中文版采用大师本人发明的 TEX 排版系统排版。我的老朋友、SAP 美国的技术咨询顾问劳佳同志在承担了排版工作的同时，亦费心费力在封面设计上帮了大忙，还对文稿内容有颇多指正之功，这里要特别致谢。我也想借此机会向在工作和生活上给了我莫大支持的父母和家人表达我内心最深处的敬意，希望本书的出版能给你们带来快乐。



2011 年 12 月

于盛大集团上海总部

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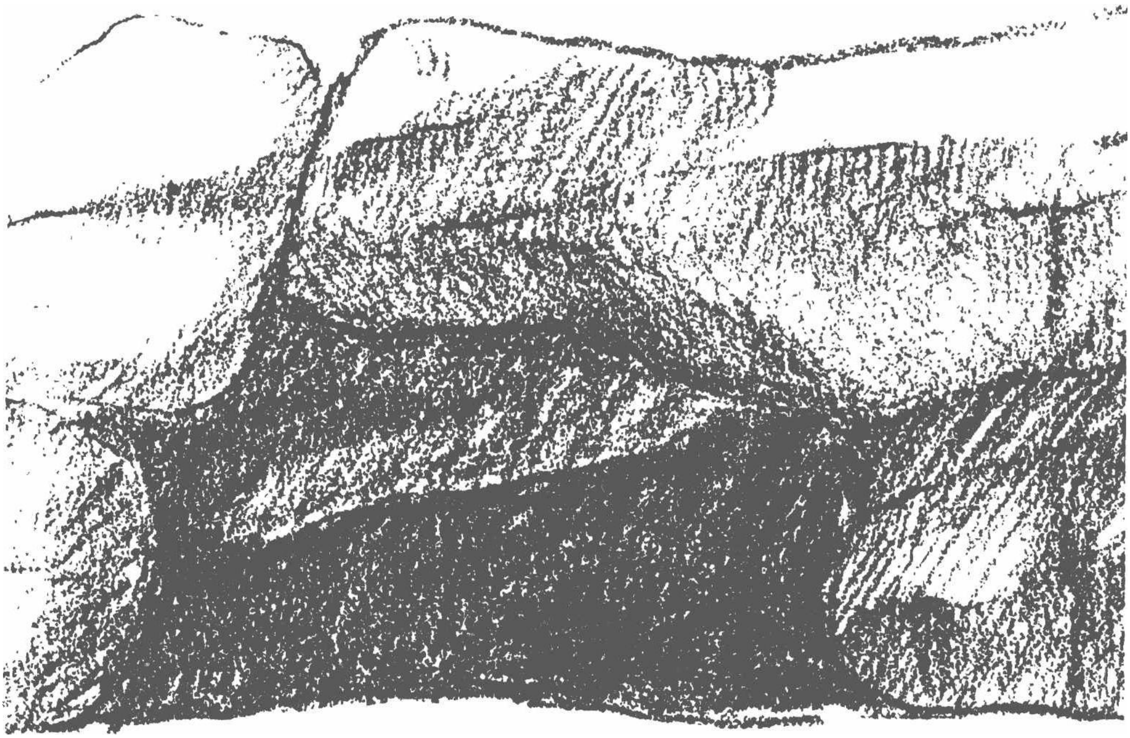


A：Bill，你觉得你已经找到自我了吗？

B：你说什么？

A：我是说——我们现在身处印度洋的边缘，远离尘嚣。为了不被世俗的体系拖垮，我们逃离尘世来“寻找自我”，到现在算起来已有好几个月。我就是想了解一下，你觉得我们已经成功地做到这一点了吗？

B. Actually, Alice, I've been thinking about the same thing. These past months together have been really great — we're completely free, we know each other, and we feel like real people again instead of like machines. But lately I'm afraid I've been missing some of the things we've "escaped" from. You know, I've got this fantastic craving for a book to read — *any* book, even a textbook, even a math textbook. It sounds crazy, but I've been lying here wishing I had a crossword puzzle to work on.

A. Oh, c'mon, not a crossword puzzle; that's what your *parents* like to do. But I know what you mean, we need some mental stimulation. It's kinda like the end of summer vacations when we were kids. In May every year we couldn't wait to get out of school, and the days simply dragged on until vacation started, but by September we were real glad to be back in the classroom.

B. Of course, with a loaf of bread, a jug of wine, and thou beside me, these days aren't exactly "dragging on." But I think maybe the most important thing I've learned on this trip is that the simple, romantic life isn't enough for me. I need something complicated to think about.

A. Well, I'm sorry I'm not complicated enough for you. Why don't we get up and explore some more of the beach? Maybe we'll find some pebbles or something that we can use to make up some kind of a game.

B. (sitting up) Yeah, that's a good idea. But first I think I'll take a little swim.

A. (running toward the water) Me, too — bet you can't catch me!

. . . . . . . .

B. Hey, what's that big black rock half-buried in the sand over there?

B：其实吧，Alice，我也一直想着这事儿呢。这几个月过得真是开心——我们彻底解放了一把，我们也了解了彼此，并且我们总算又感觉到自己活得像个人样，而不是像机器那种活法儿了。不过，最近我老是在想那些我们以前总在竭力逃避的一些东西。你也看到了，我着了魔似地想弄本书来翻翻——什么书都成，即使是教科书，甚至是数学教科书也无妨。听上去是不是我脑子有毛病了啊，但我躺在这里的时候，真希望哪怕有个纵横字谜让我填填也好啊。

A：我说，纵横字谜就算了吧，那是老一辈的人才玩的东西了。不过我知道你是什么意思，我们得来点儿脑力刺激才行。就像我们还是小孩子的时候，放暑假的感觉。一到五月，所有的小朋友们都迫不及待地想离开学校，盼着暑假的那段日子可真是难熬，可是到了九月开学，我们还是兴高采烈地回到教室的。

B：当然，如果啃着香喷喷的面包，又可以喝点儿小酒，还有佳人如你在侧，那段时间也不能说是“熬”过去的嘛。但是我觉得，这次出游给我带来的最重要的感受是简单浪漫的生活并不能满足我。我需要一些更复杂的东西来思考才行。

A：好吧，看来我对你来说原来还不够复杂啊。那你何不起来到海滩上多探探宝呢？说不定还能找到一些鹅卵石什么的，我们可以用它们想出什么新的游戏花样来。

B：（坐起身来）好啊，这个主意不赖。不过我想先游一小会儿泳再说。

A：（奔向海水）我也要——你肯定追不上我！

…………

B：看，那边半埋在沙中的黑色大块岩石样的是什么？

A. Search me, I've never seen anything like it before. Look, it's got some kind of graffiti on the back.

B. Let's see. Can you help me dig it out? It looks like a museum piece. Unnh! Heavy, too. The carving might be some old Arabian script ... no, wait, I think it's maybe Hebrew; let's turn it around this way.

A. Hebrew! Are you sure?

B. Well, I learned a lot of Hebrew when I was younger, and I can almost read this. ...

A. I heard there hasn't been much archæological digging around these parts. Maybe we've found another Rosetta Stone or something. What does it say, can you make anything out?

B. Wait a minute, gimme a chance. ... Up here at the top right is where it starts, something like "In the beginning everything was void, and ..."

A. Far out! That sounds like the first book of Moses, in the Bible. Wasn't he wandering around Arabia for forty years with his followers before going up to Israel? You don't suppose ...

B. No, no, it goes on much different from the traditional account. Let's lug this thing back to our camp, I think I can work out a translation.

A. Bill, this is wild, just what you needed!

B. Yeah, I did say I was dying for something to read, didn't I. Although this wasn't exactly what I had in mind! I can hardly wait to get a good look at it — some of the things are kinda strange, and I can't figure out whether it's a story or what. There's something about numbers, and ...

A. It seems to be broken off at the bottom; the stone was originally longer.

A：我真的不知道呀，以前还从来没有见过这样的家伙呢。你看，在它背后有一些像是涂鸦的东西呢！

B：我来看看。能帮我把它挖出来吗？这像是个老古董了。哟！还挺沉的。这些雕文像是一些阿拉伯文字……不，等等，我觉得好像是希伯来文。我们把它翻过来这样看看。

A：希伯来文！你确定吗？

B：那个，我小时候学过不少希伯来文，这上面的文字应该差不多都能看懂……

A：我并没有听说这一带有过很多考古发掘。也许我们又发现了一块罗塞塔石碑 (Rosetta Stone)[1]什么的。它上面写了什么，你能看出门道来吗？

B：等一下，让我试试……那个上面，右上角是文字开始的地方，好像是“初，万物混沌苍茫，尔后……”

A：好棒！这听起来像是《圣经》里摩西写的第一卷经书哦。那个不是摩西和他的追随者们在动身去往以色列之前在阿拉伯彷徨了四十年的事儿吗？难道……

B：不，不，这上面写的和古典经文可是大相径庭。咱们先把这个搬到帐篷里去，我想我可以把它翻译出来。

A：Bill，真不可思议啊，这正是你想要的！

B：对，我是说过我想找点儿什么看看都快想疯了，对吧。尽管这玩意儿和我脑子里想到的东西并不太一样！可我已经等不及想仔细对这东西研究研究——它看起来有点儿不寻常，我还不能断定它上面写的是个故事还是什么别的东西。好像有一些和数有关系的内容，还有……

A：这石头好像从底部断开来了，它原本更长一些的。

[1] 一块制作于公元前 196 年的大理石石碑，刻有埃及国王托勒密五世 (Ptolemy V) 的诏书。由于这块石碑刻有同一段文字的三种不同语言版本，近代的考古学家得以有机会对照内容，解读出已经失传千余年的埃及象形文的意义与结构。

B. A good thing, or we'd never be able to carry it. Of course it'll be just our luck to find out the message is getting interesting, right when we come to the broken place.

A. Here we are. I'll go pick some dates and fruit for supper while you work out the translation. Too bad languages aren't my thing, or I'd try to help you.

. . . . . . . . .

B. Okay, Alice, I've *got* it. There are a few doubtful places, a couple signs I don't recognize; you know, maybe some obsolete word forms. Overall I think I know what it says, though I don't know what it means. Here's a fairly literal translation:

> **In the beginning, everything was void, and J. H. W. H. Conway began to create numbers. Conway said, "Let there be two rules which bring forth all numbers large and small. This shall be the first rule: Every number corresponds to two sets of previously created numbers, such that no member of the left set is greater than or equal to any member of the right set. And the second rule shall be this: One number is less than or equal to another number if and only if no member of the first number's left set is greater than or equal to the second number, and no member of the second number's right set is less than or equal to the first number." And Conway examined these two rules he had made, and behold! They were very good.**
>
> **And the first number was created from the void left set and the void right set. Conway called this number "zero," and said that it shall be a sign to separate positive numbers from negative numbers. Conway proved that zero was less than or equal to zero, and he saw that it was good. And the evening and the morning were the day of zero. On the next day, two more numbers were created, one**

B：好事儿，不然我们就搬不动啦。当然，如果说我们更走运一点儿的话，那就是在看到断开的地方之前，已经开始出现一些有意义的文字了呢。

A：我们到了。我去找些枣儿水果什么的当晚饭，你就在这儿做你的翻译吧。可惜我实在不是搞语言的料，要不然我就帮你一把了。

…………

B：好了，Alice，我搞定了。当然还是有些不确定的地方，有些符号我没看明白是什么意思，有些可能是已经废弃不用的构词。大体上我已经知道它讲的是什么内容了，但还没弄明白这些内容是什么意思。下面是严格逐字逐句的翻译：

> **初，万物混沌苍茫，尔后 J. H. W. H. Conway 始创诸数。Conway 曰，"创生二道，大小诸数盖由此出。其一曰：凡数，皆合于前创二数之集，其位左者，无一大于或等于其位右者。其二曰：甲数小于或等于乙数，当且仅当甲数之左集中无一大于或等于乙数，且乙数之右集中无一小于或等于甲数。"Conway 检视二道，连呼妙哉！此二道真妙绝。**

> **元初之数，左右皆空。Conway 名之曰"零"，命其为正负两界分野之符。Conway 证得，零小于或等于零，此间妙也。夜去昼来，是为零日。次日，又得二数。其一以零为左集，**

**with zero as its left set and one with zero as its right set. And Conway called the former number "one," and the latter he called "minus one." And he proved that minus one is less than but not equal to zero and zero is less than but not equal to one. And the evening ...**

That's where it breaks off.

A. Are you *sure* it reads like that?

B. More or less. I dressed it up a bit.

A. But "Conway" ... that's not a Hebrew name. You've got to be kidding.

B. No, honest. Of course the old Hebrew writing doesn't show any vowels, so the real name might be Keenawu or something; maybe related to the Khans? I guess not. Since I'm translating into English, I just used an English name. Look, here are the places where it shows up on the stone. The J. H. W. H. might also stand for "Jehovah."

A. No vowels, eh? So it's real. ... But what do you think it means?

B. Your guess is as good as mine. These two crazy rules for numbers. Maybe it's some ancient method of arithmetic that's been obsolete since the wheel was invented. It might be fun to figure them out, tomorrow; but the sun's going down pretty soon so we'd better eat and turn in.

A. Okay, but read it to me once more. I want to think it over, and the first time I didn't believe you were serious.

B. (pointing) "In the beginning, ..."

**其一以零为右集。Conway 名前者曰"一",后者曰"负一"。Conway 证得,负一小于而不等于零,零小于而不等于一。是夜……**

岩石就是从这里断开了。

A：你确信上面写的是这些东西吗?

B：多多少少吧。我稍加了点儿修饰。

A：但那个"Conway"……这不是个希伯来名吧。你肯定是在开玩笑。

B：不,我是实话实说。当然,纸草希伯来文没有给出任何元音,所以真实的姓名可能是 Keenawu 什么的,是不是和"可汗"有点儿关系?我想大概没什么关系。因为我是把这段话译成英文的,所以我就用了个英文名。你看,这就是名字在石头上出现的位置,J. H. W. H. 可能是"Jehovah"(耶和华) 的缩写吧。

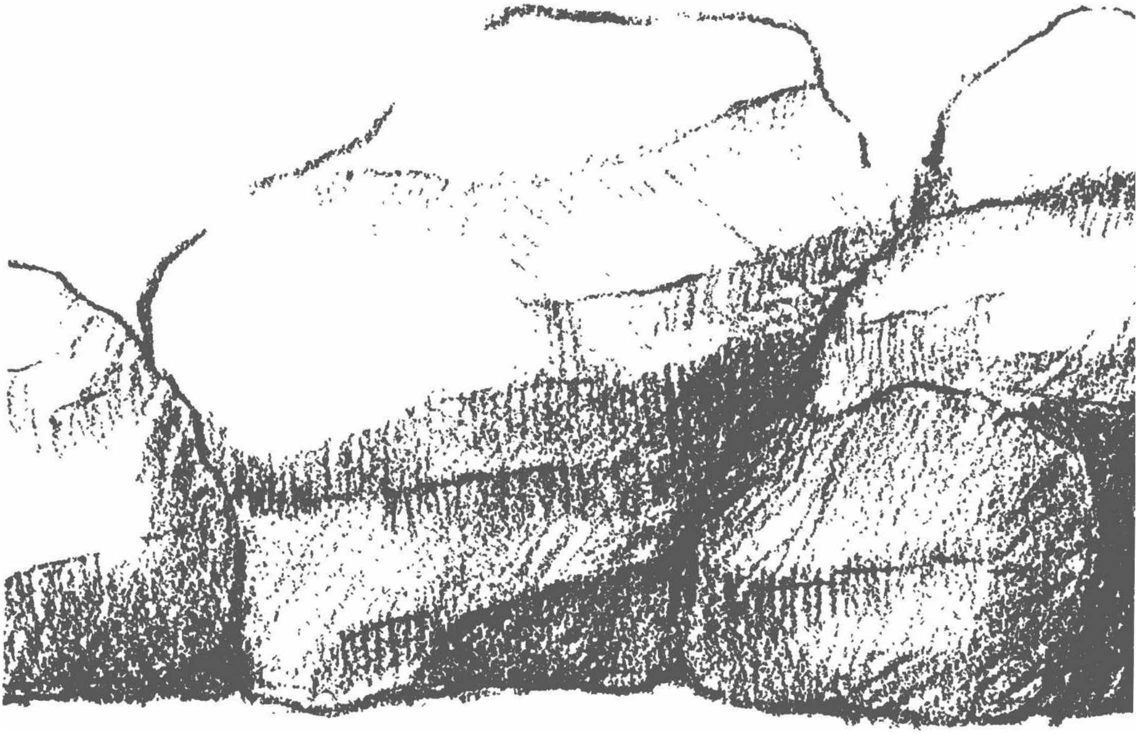
A：没有元音?呃,那真是……可你觉得这些话是什么意思呢?

B：你猜的和我猜的应该是半斤八两。这是有关数的两条不可思议的规则。也许这是古代某种算术法,老早的时候就给废弃了。把它研究出来应该会很有意思,明天吧。现在太阳马上就下山了,我们吃点东西打点歇息了吧。

A：好的,不过你再给我念一遍。我想再在脑子里过一过,这是我第一次觉得你在逗我玩儿。

B：(指点着岩石)"初,……"

# 2 SYMBOLS

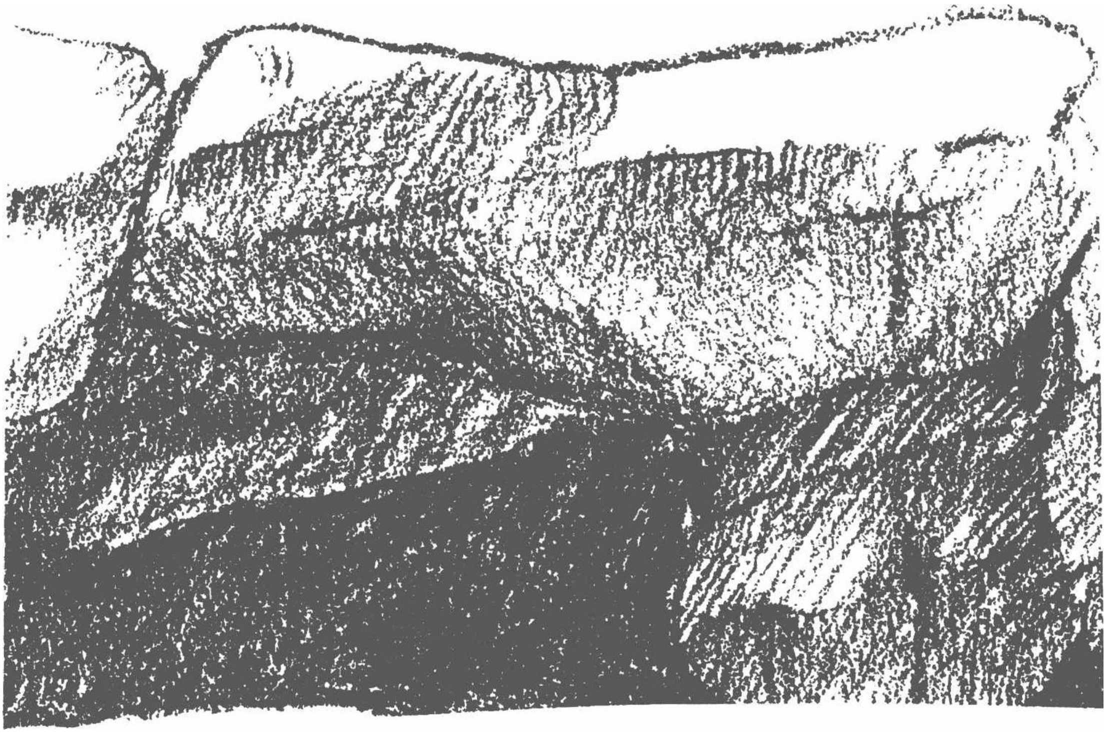


A. I think your Conway Stone makes sense after all, Bill. I was thinking about it during the night.

B. So was I, but I dozed off before getting anywhere. What's the secret?

A. It's not so hard, really; the trouble is that it's all expressed in words. The same thing can be expressed in symbols and then you can see what's happening.

# 2 符 号



A：我觉得你那"Conway 之岩"说到底还是有理可循的，Bill。我一整夜都在琢磨这个。

B：我也是啊，但是我还没琢磨明白就睡过去了。你琢磨出什么来了？

A：其实说起来也不算太难，难的是用适当的词句将它表达出来。我可以用符号来表达同样的意思，这样你一看就明白了。

B. You mean we're actually going to use the New Math to decipher this old stone tablet.

A. I hate to admit it, but that's what it looks like. Here, the first rule says that every number $x$ is really a pair of sets, called the left set $x_L$ and the right set $x_R$:

$$x = (x_L, x_R).$$

B. Wait a sec, you don't have to draw in the sand, I think we still have a pencil and some paper in my backpack. Just a minute. ... Here, use this.

A. $$x = (x_L, x_R).$$

These $x_L$ and $x_R$ are not just numbers, they're *sets* of numbers; and each number in the set is itself a pair of sets, and so on.

B. Hold it, your notation mixes me up. I don't know what's a set and what's a number.

A. Okay, I'll use capital letters for sets of numbers and small letters for numbers. Conway's first rule is that

$$x = (X_L, X_R), \qquad \text{where} \qquad X_L \not\geq X_R. \tag{1}$$

This means if $x_L$ is any number in $X_L$ and if $x_R$ is any number in $X_R$, they must satisfy $x_L \not\geq x_R$. And that means $x_L$ is not greater than or equal to $x_R$.

B. (scratching his head) I'm afraid you're still going too fast for me. Remember, you've already got this thing psyched out, but I'm still at the beginning. If a number is a pair of sets of numbers, each of which is a pair of sets of numbers, and so on and so on, how does the whole thing get started in the first place?

A. Good point, but that's the whole beauty of Conway's scheme. Each element of $X_L$ and $X_R$ must have been created previously, but on the first day of creation there weren't any previous

B：你是说，我们要用当代数学来解密这段古老石板上的文字吗？

A：我真不想承认，但好像还真的就得这么办。你瞧，这第一条规则是说，所有的数 $x$ 实际上都是一对集合，称为左集 $x_L$ 和右集 $x_R$：

$$x=(x_L,x_R)$$

B：等一等，你没必要把这些画在沙地上，我想起来我背包里还带了一支铅笔和一些纸呢，我找找……给你，用这个写。

A：$x=(x_L,x_R)$

此处，$x_L$ 和 $x_R$ 并不仅仅是数，而是数的集合。并且，这些集合中的每一个数本身也是一对集合，依此类推。

B：慢点儿，你的记法把我搞糊涂了。我分不清哪个表示集合，哪个表示数了。

A：好吧，那我就用大写字母表示数的集合，而用小写字母表示数。Conway 的第一条规则就是说：

$$x=(X_L,X_R),\quad \text{其中}\quad X_L\not\geq X_R \tag{1}$$

这就是说，若 $x_L$ 是 $X_L$ 中的任意数，而 $x_R$ 是 $X_R$ 中的任意数，则必须满足 $x_L\not\geq x_R$。而这就表示 $x_L$ 不大于也不等于 $x_R$。

B：（挠了挠脑袋）你说得有点儿太快了，我跟不上。要知道，你已经就这个问题研究了好半天了，我这才刚开始呢。如果一个数其实是一对数集，而且这些数集中的每一个数本身又都是一对数集，这样周而复始，但所有这些总得有个开始吧？

A：说得好，这正是 Conway 体系的妙处。$X_L$ 和 $X_R$ 中的每一个元素都必须“合于前创二数”，但是在诸数创生的首日，并没有任何所谓“前创”的数

numbers to work with; so both $X_L$ and $X_R$ were taken to be the empty set!

B. I never thought I'd live to see the day when the empty set was meaningful. That's really creating something out of nothing, eh? But is $X_L \not\geq X_R$ when $X_L$ and $X_R$ are both equal to the empty set? How can you have something unequal itself?

Oh yeah, yeah, that's okay since it means no *element* of the empty set is greater than or equal to any element of the empty set — it's a true statement because there *aren't* any elements in the empty set.

A. So everything gets started all right, and that's the number called zero. Using the symbol $\emptyset$ to stand for the empty set, we can write

$$0 = (\emptyset, \emptyset).$$

B. Incredible.

A. Now on the second day, it's possible to use 0 in the left or right set, so Conway gets two more numbers

$$-1 = (\emptyset, \{0\}) \qquad \text{and} \qquad 1 = (\{0\}, \emptyset).$$

B. Let me see, does this check out? For $-1$ to be a number, it has to be true that no element of the empty set is greater than or equal to 0. And for 1, it must be that 0 is not greater than any element of the empty set. Man, that empty set sure gets around! Someday I think I'll write a book called *Properties of the Empty Set*.

A. You'd never finish.

If $X_L$ or $X_R$ is empty, the condition $X_L \not\geq X_R$ is true no matter *what* is in the other set. This means that infinitely many numbers are going to be created.

可以拿来用,所以 $X_L$ 和 $X_R$ 只能都取空集!

B：没想到我在有生之年会看到空集有了意义的一天。这真是无中生有啊,对吧？但是 $X_L$ 和 $X_R$ 都是空集时,$X_L \not\geq X_R$ 成立吗？你怎么能说某物和它自己不相等呢？

哦,对的,没错儿,这一点之所以成立,是因为空集中没有元素会大于或等于空集中的任何元素——这是个真命题,因为空集中没有任何元素。

A：所以,一切各就各位,而那个数就被称为“零”。如果用符号 $\emptyset$ 表示空集,我们就得到

$$0 = (\emptyset, \emptyset)$$

B：真是妙不可言。

A：那么次日,就可以在左集或右集中使用 0 了,这样 Conway 就得到了两个新数

$$-1 = (\emptyset, \{0\}), \quad 以及 \quad 1 = (\{0\}, \emptyset)$$

B：让我瞧瞧,这些数符不符合规则？$-1$ 作为一个数,它必须满足空集中没有任何元素大于或等于 0。而对于 1 来说,它必须满足 0 不比空集中的任何元素大。哇哦,空集肯定能满足这些要求！以后我一定要写本书,名字叫做《空集的性质》。

A：你写不出来的啦!

若 $X_L$ 或 $X_R$ 为空,则另一个集合中无论有什么元素,$X_L \not\geq X_R$ 皆成立。这就意味着可以创造出无穷多个数来。

B. Okay, but what about Conway's second rule?

A. That's what you use to tell whether $X_L \not\geq X_R$, when both sets are nonempty; it's the rule defining less-than-or-equal. Symbolically,

$$x \leq y \qquad \text{means} \qquad X_L \not\geq y \qquad \text{and} \qquad x \not\geq Y_R. \tag{2}$$

B. Wait a minute, you're way ahead of me again. Look, $X_L$ is a set of numbers, and $y$ is a number, which means a pair of sets of numbers. What do you mean when you write "$X_L \not\geq y$"?

A. I mean that every element of $X_L$ satisfies $x_L \not\geq y$. In other words, no element of $X_L$ is greater than or equal to $y$.

B. Oh, I see, and your rule (2) says also that $x$ is not greater than or equal to any element of $Y_R$. Let me check that with the text. . . .

A. The Stone's version is a little different, but $x \leq y$ must mean the same thing as $y \geq x$.

B. Yeah, you're right. Hey, wait a sec, look here at these carvings off to the side:

● = ⟨:⟩

| = ⟨●:⟩

— = ⟨:●⟩

These are the symbols I couldn't decipher yesterday, and your notation makes it all crystal clear! Those double dots separate the left set from the right set. You must be on the right track.

A. Wow, equal signs and everything! That stone-age carver must have used — to stand for $-1$; I almost like his notation better than mine.

B：明白，那 Conway 的第二条规则又在说什么呢？

A：它用以在 $X_L$ 或 $X_R$ 皆不为空时，判定 $X_L \not\geq X_R$ 成立与否。这条规则定义了“小于或等于”的意义。可以用符号表示为

$$x \leq y \quad \text{意味着} \quad X_L \not\geq y \quad \text{且} \quad x \not\geq Y_R. \tag{2}$$

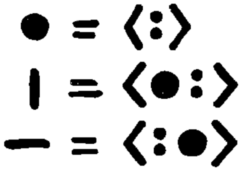
B：等一下，我又跟不上你说的速度了。看看这个，$X_L$ 是个数集，而 $y$ 是个数，也就是一对数集，那你写的 $X_L \not\geq y$ 表示什么意思呢？

A：我的意思是 $X_L$ 中的每个元素 $x_L$ 都满足 $x_L \not\geq y$。换言之，就是 $X_L$ 没有任何元素大于或等于 $y$。

B：哦，我明白了。你这里写的规则 (2) 是说，$x$ 不大于或等于 $Y_R$ 中的任何元素。让我对照这段文字检查一下……

A：岩石上的原文版本略有不同，但是，$x \leq y$ 其实和 $y \geq x$ 是一回事。

B：你说得没错。嘿，等一下，你看看边上的这些雕文：



这些符号我昨天还一头雾水，但你的记法让它们变得一目了然啦！这个双点符隔开了左集和右集。你肯定已经在正轨上了。

A：哇哦，等于号，以及所有的这一切！石器时代的人们一定是用 ━ 来代表 $-1$。比起自己的符号，我简直还更喜欢他们的。

B. I bet we've underestimated primitive people. They must have had complex lives and a need for mental gymnastics, just like us—at least when they didn't have to fight for food and shelter. We always oversimplify history when we look back.

A. Yes, but otherwise how could we look back?

B. I see your point.

A. Now comes the part of the text I don't understand. On the first day of creation, Conway "proves" that $0 \leq 0$. Why should he bother to prove that something is less than or equal to itself, since it's obviously equal to itself. And then on the second day he "proves" that $-1$ is not equal to 0; isn't that obvious without proof, since $-1$ is a different number?

B. Hmmm. I don't know about you, but I'm ready for another swim.

A. Good idea. That surf looks good, and I'm not used to so much concentration. Let's go!

B：我敢打赌，我们是低估原始人了。他们一定和我们一样，有过高层次的生活，而且也需要思维的体操 ——至少在他们衣食无忧时会有。我们在回顾历史时总是会过分简化。

A：不然你怎么个回顾法儿？

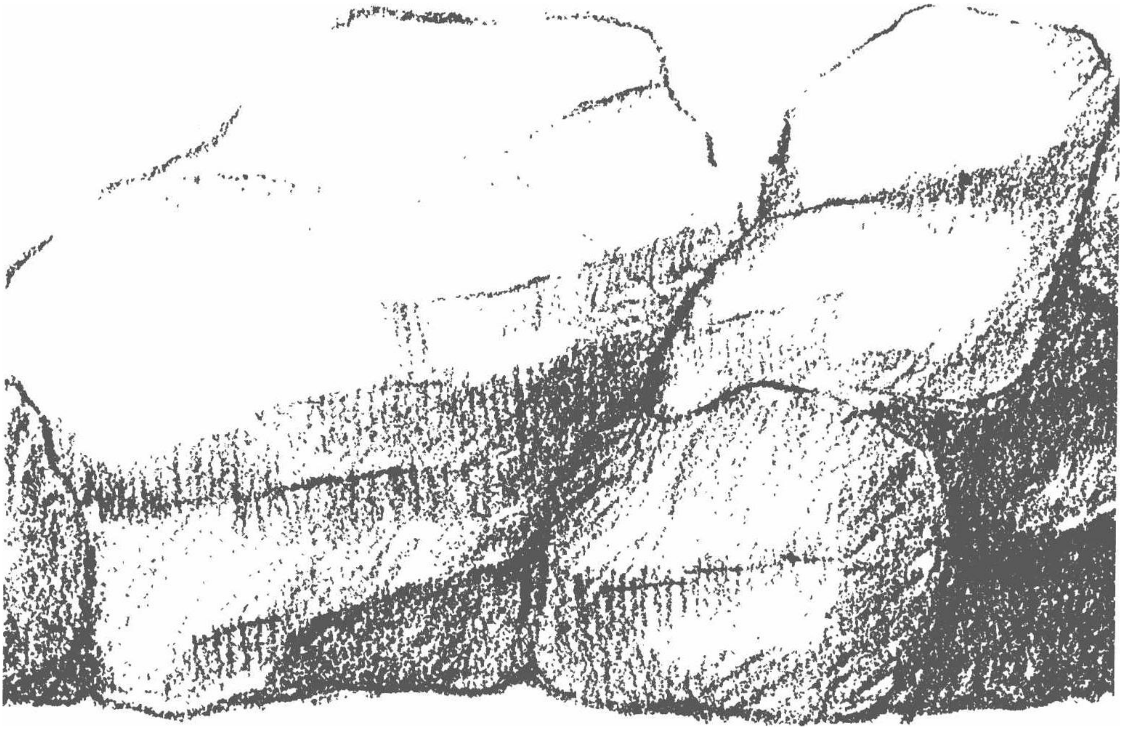
B：我明白你的意思。

A：下面这段话我可就看不懂了。在诸数创生的首日，Conway“证明”了 $0 \leq 0$。为什么他要费神去证明某数小于或等于它自身？而在次日，他“证明”了 $-1$ 不等于 0，而 $-1$ 本来就是另一个数，这难道还用证？

B：嗯，这个嘛。我不知道你是怎么想的，反正我现在又想游会儿泳了。

A：好呀。这会儿冲浪条件看起来不错，而且我也不习惯集中精力那么久。走，一起吧！

# 3 PROOFS

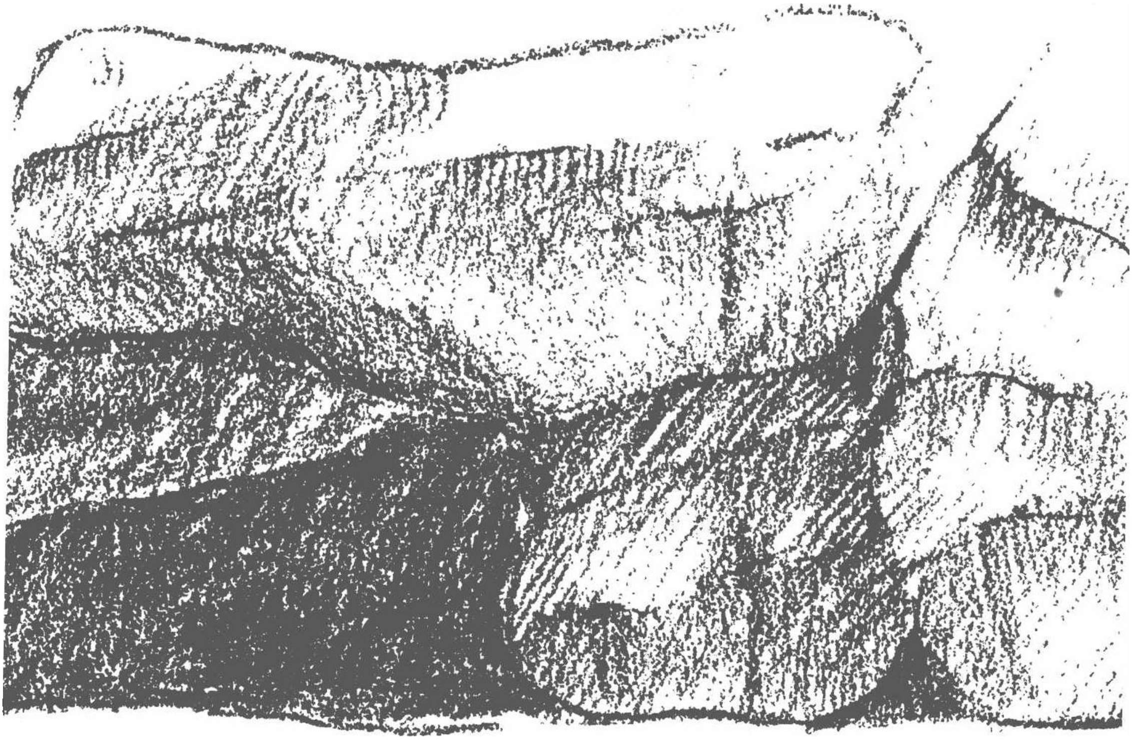


B. An idea hit me while we were paddling around out there. Maybe my translation *isn't* correct.

A. What? It *must* be okay, we've already checked so much of it out.

B. I know; but now that I think of it, I wasn't quite sure of that word I translated "equal to." Maybe it has a weaker meaning, "similar to" or "like." Then Conway's second rule

# 3 证　明



B：刚才咱们在那边划水的时候，我忽然闪过一个念头，想到我的翻译可能有个地方并不正确。

A：你说什么？你那翻译肯定是八九不离十的，我们已经推导出这么多结果来了。

B：我明白，但转念一想，我就有点儿拿不准我译成"等于"的词儿了。可能它只有较弱的意义，也就是"相似于"或是"相像"那个程度。这么一来，Conway 的第二条规则

becomes "One number is less than or *like* another number if and only if ... ." And later on, he proves that zero is less than or *like* zero; minus one is less than but not like zero; and so forth.

A. Oh, right, that must be it, he's using the word in an abstract technical sense that must be defined by the rules. So of *course* he wants to prove that 0 is less than or like 0, in order to see that his definition makes a number "like" itself.

B. So does his proof go through? By rule (2), he must show that no element of the empty set is greater than or like 0, and that 0 is not greater than or like any element of the empty set. ... Okay, it works, the empty set triumphs again.

A. More interesting is how he could prove that $-1$ is *not* like 0. The only way I can think of is that he proved that 0 is not less-than-or-like $-1$. I mean, we have rule (2) to tell whether one number is less than or like another; and if $x$ is not less-than-or-like $y$, it isn't less than $y$ and it isn't like $y$.

B. I see, we want to show that $0 \leq -1$ is false. This is rule (2) with $x = 0$ and $Y_R = \{0\}$, so $0 \leq -1$ if and only if $0 \ngeq 0$. But 0 *is* $\geq 0$, we know that, so $0 \nleq -1$. He was right.

A. I wonder if Conway also tested $-1$ against 1; I suppose he did, although the rock doesn't say anything about it. If the rules are any good, there should be a way to prove that $-1$ is less than 1.

B. Well, let's see: $-1$ is $(\emptyset, \{0\})$ and 1 is $(\{0\}, \emptyset)$, so once again the empty set makes $-1 \leq 1$ by rule (2). On the other hand, $1 \leq -1$ is the same as saying that $0 \ngeq -1$ and $1 \ngeq 0$, according to rule (2), but we know that both of these are false. Therefore $1 \nleq -1$, and it must be that $-1 < 1$. Conway's rules seem to be working.

A. Yes, but so far we've been using the empty set in almost every argument, so the full implications of the rules aren't clear yet.

就变成了“甲数小于或相似于乙数，当且仅当……”再后面，他证明了零小于或相似于零，负一小于而不相似于零，依此类推。

A：哦，没错，肯定是这样了，他在使用这个词的时候，实际上是采取了一种抽象观念，而这种抽象观念必须用法则来加以定义。所以，理所当然地，他需要证明 0 小于或相似于 0，这样才可以确定在他的定义中一个数相似于它本身。

B：那末，他的证明行得通吗？由规则 (2)，他必须证明空集中没有元素大于或相似于 0，且 0 不大于或相似于空集中的任何元素……好吧，这显然成立，空集又赢了。

A：更意味深长的是，他可以证明 $-1$ 不相似于 0。要做到这一点，我能想到的唯一办法是他要去证明 0 不小于也不相似于 $-1$。我是说，既然由规则 (2) 可以判定某数是否小于或相似于另一个数，那末若 $x$ 小于或相似于 $y$ 不成立，则 $x$ 不小于 $y$，且 $x$ 不相似于 $y$。

B：我明白了，我们要证明 $0 \le -1$ 不成立。也就是在 $x = 0$ 和 $Y_R = \{0\}$ 时，应用规则 (2)，亦即 $0 \le -1$ 当且仅当 $0 \not\ge 0$。但是显然有$0 \ge 0$，这是已知的。所以 $0 \not\le -1$ 成立，他的证明没错。

A：我想知道 Conway 有没有也对 $-1$ 和 1 的关系也做过同样的检验，我认为他肯定做了，尽管岩石上没有提及此事。如果这些规则真的管用，那肯定能找出办法来证明 $-1$ 小于 1。

B：嗯，咱们来试试：$-1$ 就是 $(\emptyset, \{0\})$，而 1 就是 $(\{0\}, \emptyset)$，我们可以用空集定义，借助规则 (2) 来证得 $-1 \le 1$。反过来说，由规则 (2)，说 $1 \le -1$ 成立等于是说 $0 \not\ge -1$ 和 $1 \not\ge 0$，但我们已知后面两个命题不成立。因此 $1 \not\le -1$，即 $-1 < 1$。Conway 的规则看起来的确管用。

A：对，可是直到目前，我们好像几乎在每一次论证时都用到了空集，所以这些规则完全的涵义我们尚不清楚。

Have you noticed that almost everything we've proved so far can be put into a framework like this: "If $X$ and $Y$ are any sets of numbers, then $x = (\emptyset, X)$ and $y = (Y, \emptyset)$ are numbers, and $x \le y$."

B. It's neat the way you've just proved infinitely many things, by looking at the pattern I used in only a couple of cases. I guess that's what they call abstraction, or generalization, or something. But can you also prove that your $x$ is strictly *less* than $y$? That was true in all the simple cases and I bet it's true in general.

A. Uh huh ... Well no, not when $X$ and $Y$ are both empty, since that would mean $0 \not\geq 0$. But otherwise it looks very interesting. Let's look at the case when $X$ is the empty set and $Y$ is not empty; is it true that 0 is less than $(Y, \emptyset)$?

B. If so, then I'd call $(Y, \emptyset)$ a "positive" number. That must be what Conway meant by zero separating the positive and negative numbers.

A. Yes, but look. According to rule (2), we will have $(Y, \emptyset) \le 0$ if and only if no member of $Y$ is greater than or like 0. So if, for example, $Y$ is the set $\{-1\}$, then $(Y, \emptyset) \le 0$. Do you want positive numbers to be $\le 0$?

Too bad I didn't take you up on that bet.

B. Hmm. You mean $(Y, \emptyset)$ is going to be positive only when $Y$ contains some number that is zero or more. I suppose you're right. But at least we now understand everything that's on the stone.

A. Everything up to where it's broken off.

B. You mean ...?

A. I wonder what happened on the *third* day.

你发现了没有，我们到目前为止所证明的一切都可以套用同一个框架："若 $X$ 和 $Y$ 为任意数集，则 $x=(\emptyset,X)$ 和 $y=(Y,\emptyset)$ 为数，且 $x\le y$。"

B：真的是很简明扼要的办法，你只是观察了一下我举的几个例子，就推出了无穷多种情形。我觉得这就是人们讲的抽象、概括之类的行为吧。但是，你能否证明你这里的 $x$ 是严格地小于$y$ 的？这个命题在所有的平凡案例中都成立，我打赌它应该在一般情况下也都成立。

A：呃……好吧，不过，在 $X$ 和 $Y$ 皆为空集时就并非如此，因为那样一来就变成 $0\not\ge 0$ 了。不过在其他情形中，就很有意思了。我们先看一下 $X$ 为空集，而 $Y$ 不为空集的情形，这里 0 小于 $(Y,\emptyset)$ 成立吗？

B：如果成立的话，那我就会将 $(Y,\emptyset)$ 称作"正"数。这肯定是 Conway 以零为"正负两界分野之符"想要表达的意思。

A：话是没错，但是你瞧。由规则 (2) 我们得到，$(Y,\emptyset)\le 0$ 当且仅当 Y 中没有元素大于或相似于 0。所以，假设，举例来说，$Y$ 表示集合 $\{-1\}$ 的话，可就有 $(Y,\emptyset)\le 0$ 了。你想要正数 $\le 0$ 成立吗？真后悔当时没跟你商量打赌的赌注。

B：唔。你是说 $(Y,\emptyset)$ 是正数这个命题仅在 $Y$ 包含一些或为零、或比零大的数时才成立。我觉得你是对的。但至少我们现在已经理解石板上的所有内容了。

A：只能说是理解了所有在它断开之前的内容。

B：你的意思是……

A：我想知道在第三日发生了什么事。

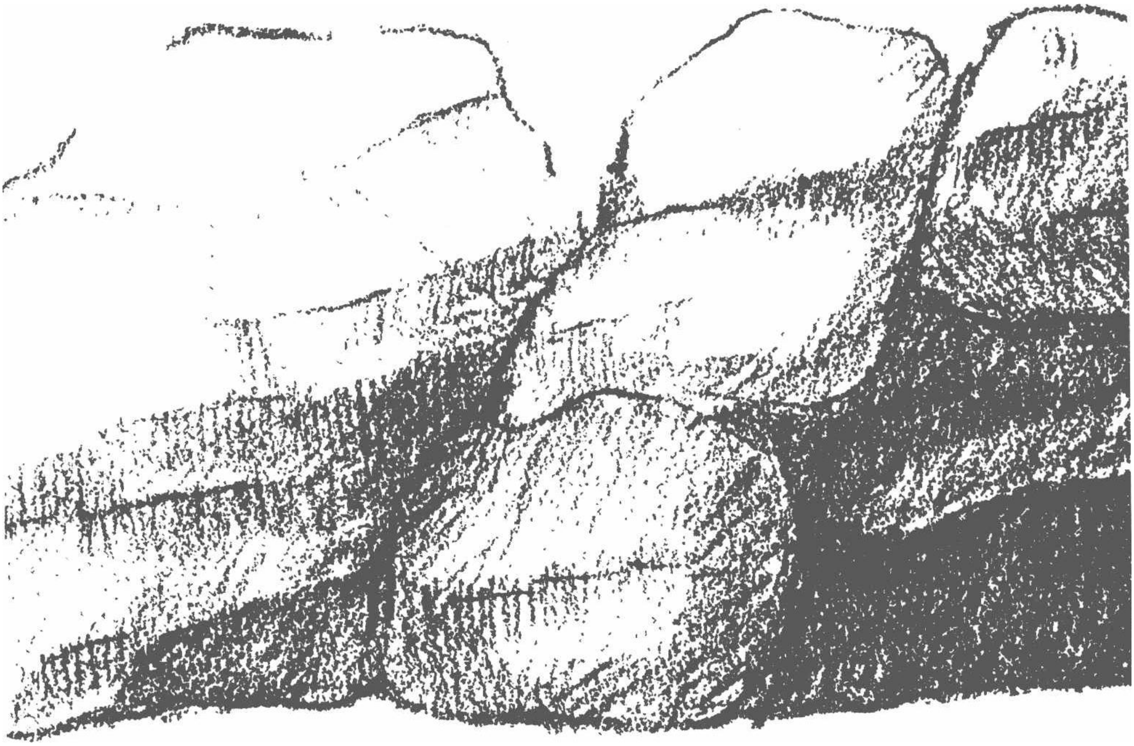
**B.** Yes, we should be able to figure that out, now that we know the rules. It might be fun to work out the third day, after lunch.

**A.** You'd better go catch some fish; our supply of dried meat is getting kinda low. I'll go try and find some coconuts.

B：这个应该可以推得，因为我们已经掌握规则了。推算出第三日的情形肯定有趣，我们吃完午饭再说吧。

A：你最好去抓几条鱼来，咱们的干肉储备可是一天比一天见少了。我去试着找几个椰子来。

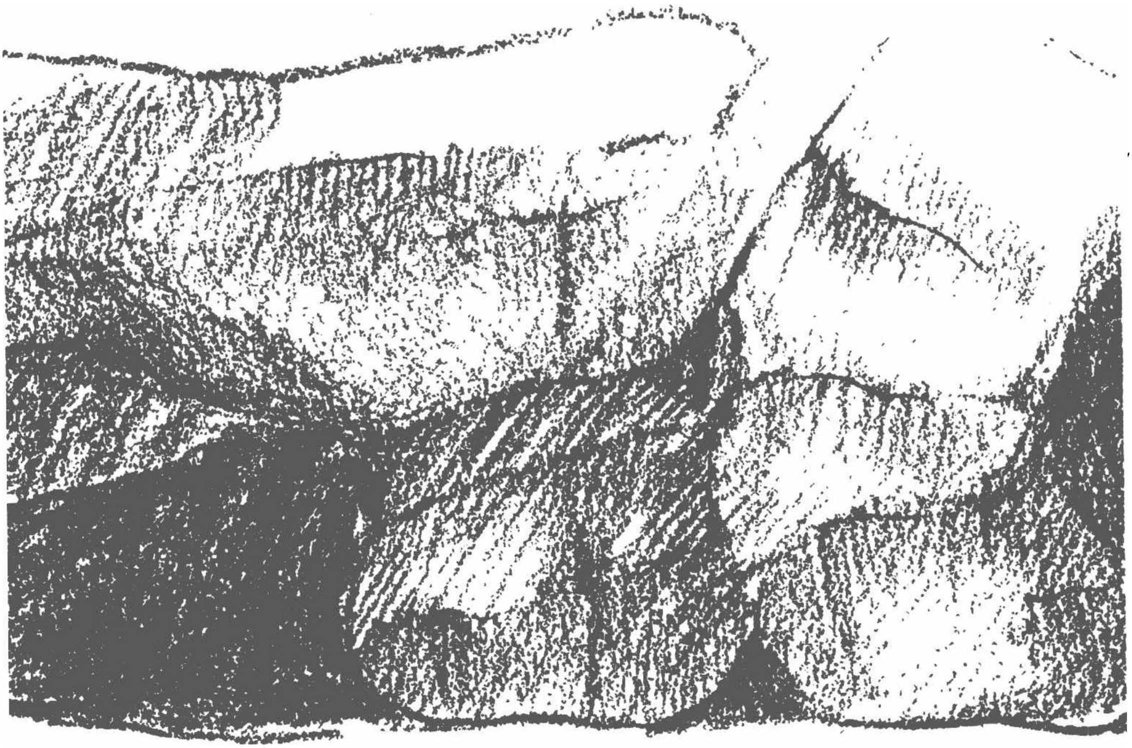
# 4 BAD NUMBERS



B. I've been working on that Third Day problem, and I'm afraid it's going to be pretty hard. When more and more numbers have been created, the number of possible sets goes up fast. I bet that by the seventh day, Conway was ready for a rest.

A. Right. I've been working on it too and I get seventeen numbers on the third day.

# 4 坏 数

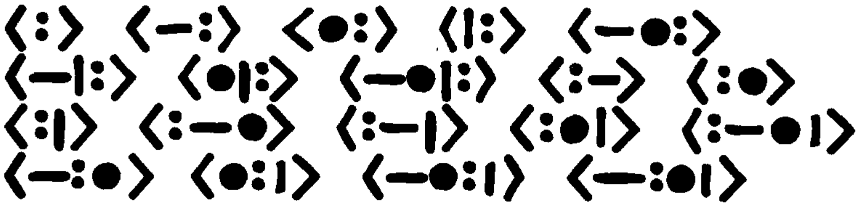


B：我一直在琢磨那个第三日的问题，我觉得那怕是比较棘手。当更多的数被创造出来的同时，数集的可能状态数目增长非常快。我敢打赌到了第七日，Conway 也该收手歇息了。[1]

A：没错，我也一直在琢磨这个问题哩，而且我推算出在第三日可以得到十七个数。

[1] 见作者序注 2。

B. Really? I found nineteen; you must have missed two. Here's my list:



A. I see you're using the Stone's notation. But why did you include ⟨:⟩? That was created already on the first day.

B. Well, we have to test the new numbers against the old, in order to see how they fit in.

A. But I only considered *new* numbers in my list of seventeen, so there must actually be *twenty* different at the end of the third day. Look, you forgot to include

in your list.

B. (blinking) So I did. Hmm ... 20 by 20, that's 400 different cases we'll have to consider in rule (2). A lot of work, and not much fun either. But there's nothing else to do, and I know it'll bug me until I know the answer.

A. Maybe we'll think of some way to simplify the job once we get started.

B. Yeah, that would be nice. ...

Well, I've got one result, 1 is less than $(\{1\}, \emptyset)$. First I had to prove that $0 \not\geq (\{1\}, \emptyset)$.

A. I've been trying a different approach. Rule (2) says we have to test every element of $X_L$ to see that it isn't greater than or like $y$, but it shouldn't be necessary to make so many tests. If

B：真的吗？我得到了十九个，你肯定是漏掉了两个。这是我得到的数列成的表：

⟨:⟩ ⟨—:⟩ ⟨●:⟩ ⟨|:⟩ ⟨—●:⟩
⟨—|:⟩ ⟨●|:⟩ ⟨—●|:⟩ ⟨:—⟩ ⟨:●⟩
⟨:|⟩ ⟨:—●⟩ ⟨:—|⟩ ⟨:●|⟩ ⟨:—●|⟩
⟨—:●⟩ ⟨●:|⟩ ⟨—●:|⟩ ⟨—:●|⟩

A：我明白，你用的是岩石上的记法。但是为什么把 ⟨:⟩ 也包括进去呢？这个数不是在首日就创造出来了吗？

B：嗯，我们必须经由旧数来验证新数，以此来决定它们应该填到什么位置。

A：但是我在我的十七个数构成的数表中只考虑了新数，所以在第三日结束时，肯定总共有二十个不同的数了。瞧，你把这个数漏掉了。

⟨—:|⟩

B：（眨巴眨巴眼睛）嗯，我还真是把这个给忘了。这么一来……20 乘以 20，根据规则 (2)，总共就有 400 种不同的情况需要我们去考虑了。这里面有大量的工作要做，也没啥乐趣可言。但是在这里又没有别的事情可做，而且我要是不能了解答案的话，心里会一直不舒服的。

A：也许我们在动手之前就得想点儿办法，来简化这个工作。

B：是啊，能简化的话就好了……

好，我已经得到了一个结果，1 小于 $(\{1\},\emptyset)$。首先我要证明的是 $0 \not\gtrsim (\{1\},\emptyset)$。

A：我是从另外一个角度来考虑的。规则 (2) 说，我们必须检测 $X_L$ 中的每一个元素是否不大于或相似于 $y$，但实际上并不需要做那么多检测。如果

any element of $X_L$ is $\geq y$, then the *largest* element of $X_L$ ought to be $\geq y$. Similarly, we need only test $x$ against the *smallest* element of $Y_R$.

B. Yeah, that oughta be right. ... I can prove that 1 is less than $(\{0,1\},\emptyset)$ just like I proved it was less than $(\{1\},\emptyset)$; the extra "0" in $X_L$ didn't seem to make any difference.

A. If what I said is true, it will save us a lot of work, because each number $(X_L, X_R)$ will behave in all $\leq$ relations exactly as if $X_L$ were replaced by its largest element and $X_R$ by its smallest. We won't have to consider any numbers in which $X_L$ or $X_R$ have two or more elements; ten of those twenty numbers in the list will be eliminated!

B. I'm not sure I follow you, but how on earth can we prove such a thing?

A. What we seem to need is something like this:

$$\text{if} \quad x \leq y \quad \text{and} \quad y \leq z, \quad \text{then} \quad x \leq z. \tag{T1}$$

I don't see that this follows immediately, although it is consistent with everything we know.

B. At any rate, it ought to be true, if Conway's numbers are to be at all decent. We could go ahead and assume it, but it would be neat to show once and for all that it is true, just by using Conway's rules.

A. Yes, and we'd be able to solve the Third Day puzzle without much more work. Let's see, how can it be proved? ...

B. Blast those flies! Just when I'm trying to concentrate. Alice, can you — no, I guess I'll go for a little walk.

. . . . . . . . .

Any progress?

$X_L$ 中的任意元素都满足 $\geq y$，则 $X_L$ 中的最大元素就应该满足 $\geq y$。相似地，我们只需要将 $x$ 与 $Y_R$ 中的最小元素比较大小即可。

B：嗯，这应该没错儿……我可以证明 1 小于 $(\{0,1\},\emptyset)$，就像我刚才证明 1 小于 $(\{1\},\emptyset)$ 那样。那个在 $X_L$ 中多出来的 0 似乎没有造成任何结果上的不同。

A：如果我所说的方法成立，那真的可以省不少事儿，因为 $(X_L, X_R)$ 在 $\leq$ 关系下的表现，与将 $X_L$ 替换为其最大元素、$X_R$ 替换为其最小元素时一模一样。这么一来，我们就用不着考虑那些 $X_L$ 或 $X_R$ 包含两个或两个以上元素的任何数了。这么一来，现有数表中的二十个数中，一下子能去掉十个！

B：我不确定是不是跟上了你的思路，但是到底怎样才能证明这么一件事呢？

A：我们所需要的，看起来是类似于这样的论断：

$$\text{若} \quad x \leq y \quad \text{且} \quad y \leq z, \quad \text{则} \quad x \leq z. \qquad \text{(T1)}$$

我还看不出如何能够立即推得这个结论，但它与我们已知的知识是相一致的。

B：无论如何，这个都应该成立，我是说如果 Conway 的数概念从根本意义上合理的话。我们当然可以不管三七二十一，直接假定它成立，但还是仅使用 Conway 的规则一次性地证明它对所有情况都成立，会更合理一些。

A：那肯定啊，这么一来解决第三日的问题也就不需要花费太多额外的力气了。那么，这个该怎么证明呢……

B：该死的苍蝇！我刚想集中精力就飞来捣乱。Alice，你能不能……呵呵，我想我得出去溜哒一圈儿。

…………

你搞出点儿名堂了吗？

A. No, I seem to be going in circles, and the $\ngtr$ versus $\le$ is confusing. Everything is stated negatively and things get incredibly tangled up.

B. Maybe (T1) isn't true.

A. But it *has* to be true. Wait, that's it! We'll try to *disprove* it. And when we fail, the cause of our failure will be a proof!

B. Sounds good — it's always easier to prove something wrong than to prove it right.

A. Suppose we've got three numbers $x$, $y$, and $z$ for which

$$x \le y, \qquad \text{and} \qquad y \le z, \qquad \text{and} \qquad x \not\le z.$$

What does rule (2) tell us about "bad numbers" like this?

B. It says that

$$\begin{aligned} & X_L \ngtr y, \\ \text{and} \quad & x \ngtr Y_R, \\ \text{and} \quad & Y_L \ngtr z, \\ \text{and} \quad & y \ngtr Z_R, \end{aligned}$$

and then also $x \not\le z$, which means what?

A. It means one of the two conditions fails. Either there is a number $x_L$ in $X_L$ for which $x_L \ge z$, or there is a number $z_R$ in $Z_R$ for which $x \ge z_R$. With all these facts about $x$, $y$, and $z$, we ought to be able to prove *something*.

B. Well, since $x_L$ is in $X_L$, it can't be greater than or like $y$. Say it's less than $y$. But $y \le z$, so $x_L$ must be ... no, sorry, I can't use facts about numbers we haven't proved.

Going the other way, we know that $y \le z$ and $z \le x_L$ and $y \not\le x_L$; so this gives us three more bad numbers, and we can get more facts again. But that looks hopelessly complicated.

A：没有，我觉得好像在兜圈子。又是 $\ngeq$ 又是 $\leq$ 的，搞得我十分头大。而且所有的内容都是负向表述的，简直一团糟啦！

B：也许 (T1) 并不成立呢？

A：但是它肯定成立。等一下，我明白了！我们先试着证伪它。如果我们发现无法证伪，那么导致这个结果的原因即为所证！

B：听上去不错 ——证明某个命题是错的，总比证明它是对的要容易吧。

A：设有三数 $x$、$y$ 和 $z$ 满足

$$x \leq y, \quad 且 \quad y \leq z, \quad 且 \quad x \nleq z.$$

规则 (2) 有没有就这种"坏数"让我们讨论呢？

B：它只指明了

$$\begin{aligned} & X_L \ngeq y, \\ 且 \quad & x \ngeq Y_R, \\ 且 \quad & Y_L \ngeq z, \\ 且 \quad & y \ngeq Z_R, \end{aligned}$$

并且还有 $x \nleq z$，这说明什么呢？

A：这意味着两个条件中的有一个不成立。不是 $X_L$ 中的某个元素 $x_L$ 满足了 $x_L \geq z$，就是 $Z_R$ 中的某个元素 $z_R$ 满足了 $x \geq z_R$。在有了关于 $x$、$y$ 和 $z$ 的这么多题设的前提之下，我们应该能够证明出一些什么吧。

B：嗯，$x_L$ 既然是 $X_L$ 中的某个元素，它就不能大于或相似于 $y$，即它小于 $y$。而又 $y \leq z$，所以 $x_L$ 肯定……不，不好意思，我不能利用我们还没有证明过的数的性质。

换个思路，我们有 $y \leq z$，且 $z \leq x_L$，且 $y \nleq x_L$。这么一来我们又得到了三个坏数，并且我们还可以重复这个过程以得到更多的条件。但这看起来真是复杂到了令人绝望的程度。

A. Bill! You've got it.

B. Have I?

A. If $(x, y, z)$ are three bad numbers, there are two possible cases.

*Case 1*, some $x_L \geq z$: Then $(y, z, x_L)$ are three more bad numbers.

*Case 2*, some $z_R \leq x$: Then $(z_R, x, y)$ are three more bad numbers.

B. But aren't you still going in circles? There's more and more bad numbers all over the place.

A. No, in each case the new bad numbers are *simpler* than the original ones; one of them was created earlier. We can't go on and on finding earlier and earlier sets of bad numbers, so there can't be any bad sets at all!

B. (brightening) Oho! What you're saying is this: Each number $x$ was created on some day $d(x)$. If there are three bad numbers $(x, y, z)$, for which the sum of their creation days is $d(x) + d(y) + d(z) = n$, then one of your two cases applies and gives three bad numbers whose day-sum is less than $n$. Those, in turn, will produce a set whose day-sum is still less, and so on; but that's impossible since there are no three numbers whose day-sum is less than 3.

A. Right, the sum of the creation days is a nice way to express the proof. If there are no three bad numbers $(x, y, z)$ whose day-sum is less than $n$, the two cases show that there are none whose day-sum equals $n$. I guess it's a proof by induction on the day-sum.

B. You and your fancy words. It's the *idea* that counts.

A. True; but we need a name for the idea, so we can apply it more easily next time.

B. Yes, I suppose there will be a next time. ...

A：Bill！你成功啦。

B：我哪有啊？

A：设 $(x,y,z)$ 是三个坏数，则有两种可能情况。

情况 *1*，某些 $x_L \geq z$：这种情况下，$(y,z,x_L)$ 又成为三个坏数。

情况 *2*，某些 $z_R \leq x$：这种情况下，$(z_R,x,y)$ 又成为三个坏数。

B：可是即便这样，你不是仍然在兜圈子吗？这样会产生越来越多的坏数，越来越乱了。

A：并非如此，两种情况中的每一种，新的坏数都比原先的更简单，并且其中一个比其他两个更先创造出来。我们寻找坏数的重复过程是不能像这样永无止境地进行下去的，所以坏数集合根本就不存在！

B：（眼前一亮）哦！你想说的是：每个数 $x$ 都是在某一日 $d(x)$ 被创造出来的。如果存在三个坏数 $(x,y,z)$，它们的创造日之和为 $d(x)+d(y)+d(z)=n$，然后根据你说的两种情况之一，得到三个坏数的创造日之和小于 $n$。而这个结果又会产生一个新的创造日之和更小的集合，依此进行。但是，这里面存在矛盾，因为不存在这样的三个数，使得它们的创造日之和小于 3。

A：对，通过创造日之和来证明是一种很妙的方法。若没有三个坏数 $(x,y,z)$ 的创造日之和小于 $n$，则可能的两种情况说明，也没有创日数之和等于 $n$ 的坏数能够满足题设。我猜想这应该是一种基于创造日之和的归纳法来证明的。

B：你说得太棒了。这就是它得以运作的思想。

A：没错，但我们需要为这个思想起个名字，这样下次就可以直接应用它了。

B：对，我相信它一定还会被再次用到的……

Okay, I guess there's no reason for me to be uptight any more about the New Math jargon. You know it and I know it, we've just proved the *transitive law.*

A. (sigh) Not bad for two amateur mathematicians!

B. It was really your doing. I hereby proclaim that the transitive law (T1) shall be known henceforth as Alice's Theorem.

A. C'mon. I'm sure Conway discovered it long ago.

B. But does that make your efforts any less creative? I bet every great mathematician started by rediscovering a bunch of "well known" results.

A. Gosh, let's not get carried away dreaming about greatness! Let's just have fun with this.

好了，我觉得不必为这个当代数学的行话而绞尽脑汁了。你我都知道这是什么，我们刚才证明的是传递律。

A：（叹了口气）对于两个业余数学家来说，这也就颇为不易了！

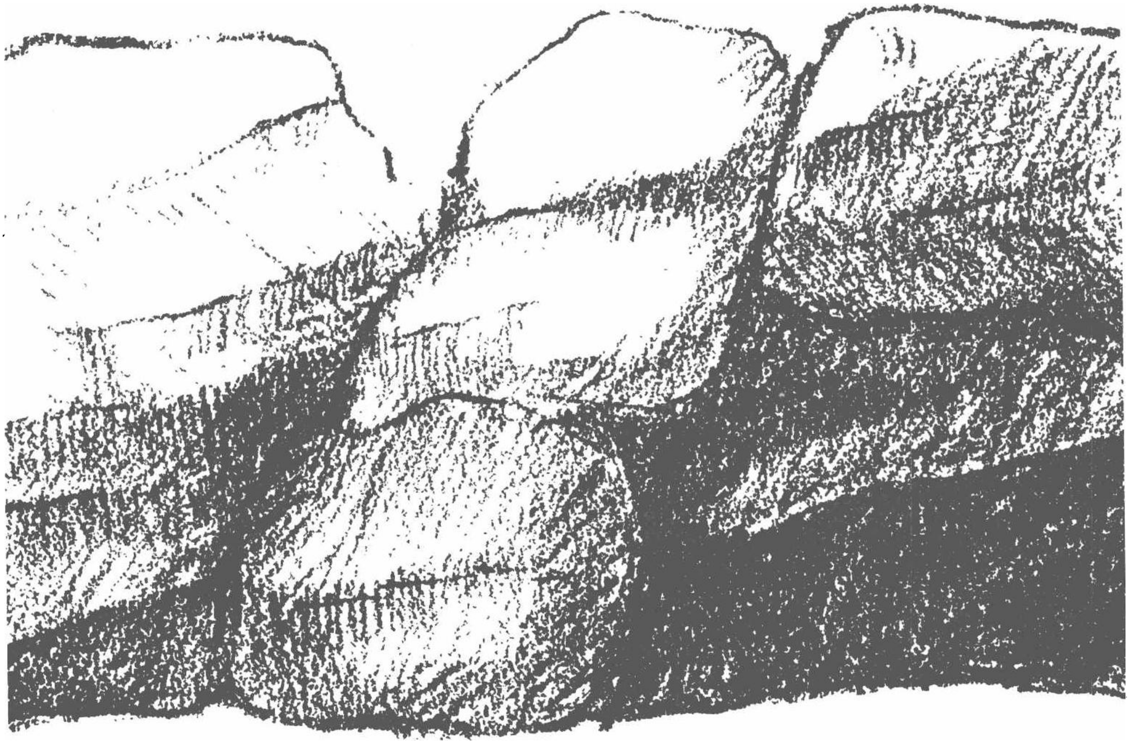
B：这都是你的功劳。在这里我隆重宣布，传递律 (T1) 今后将被称为 Alice 定理。

A：得了吧。Conway 肯定早就发现这个定理了。

B：但是那对你探索的创新性有丝毫的减损吗？我敢说任何伟大的数学家都是从重新发现一堆“众所周知”的结果起步的。

A：啊呀，我们就不要被这种成为伟人的白日梦弄得神魂颠倒啦。我们只是从这里面找些乐子罢了。

# 5 PROGRESS

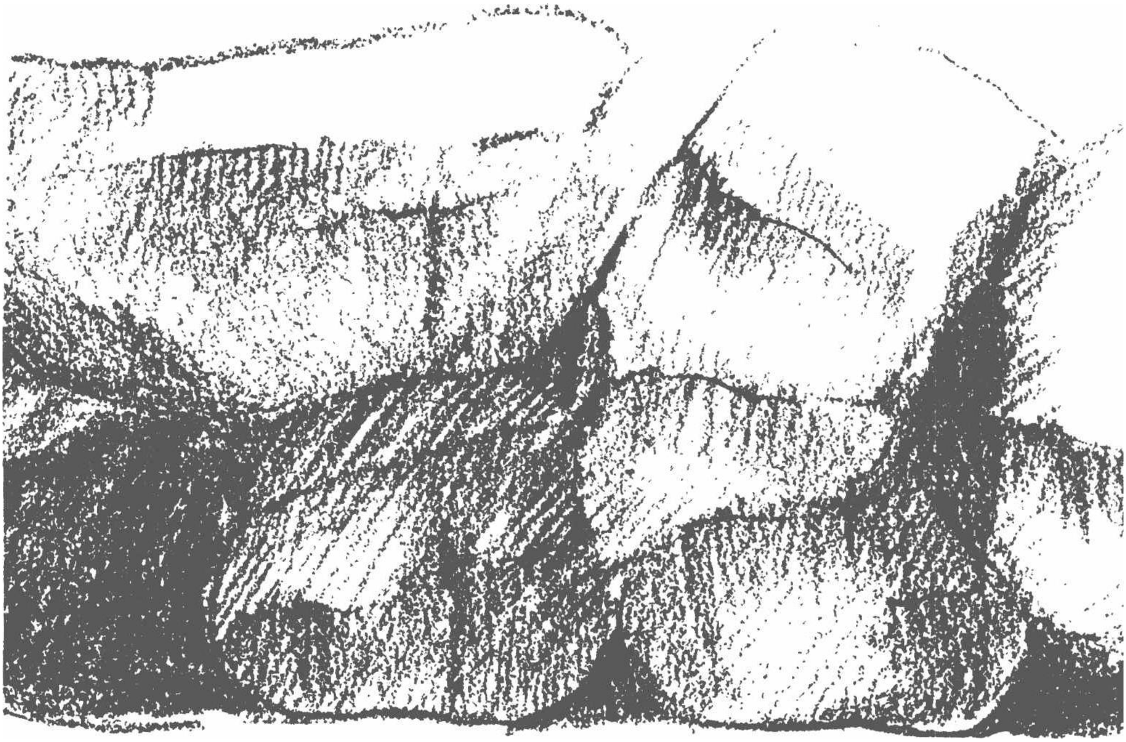


B. I just thought of something. Could there possibly be two numbers that aren't related to each other at all? I mean

$$x \not\leq y \qquad \text{and} \qquad y \not\leq x,$$

like one of them is out of sight or in another dimension or something. It shouldn't happen, but how would we prove it?

# 5 进展



B：我刚刚想到一件事。有没有可能存在两个彼此毫无关系的数？我的意思是

$$x \not\leq y \quad 且 \quad y \not\leq x,$$

就好像一个数对另一个视而不见，仿佛身处异次元空间似的。这种情况理应是不存在的，但我们又该如何证明它的确不存在呢？

A. I suppose we could try the same technique that worked before. If $x$ and $y$ are bad numbers in this sense, then either some $x_L \geq y$ or $x \geq$ some $y_R$.

B. Hmm. Suppose $y \leq x_L$. Then if $x_L \leq x$, we would have $y \leq x$ by our transitive law, and we have assumed that $y \not\leq x$. So $x_L \not\leq x$. In the other case, $y_R \leq x$, the same kind of figuring would show that $y \not\leq y_R$.

A. Hey, that's very shrewd! All we have to do now, to show that such a thing can't happen, is prove something I've suspected for a long time. Every number $x$ must lie between all the elements of its sets $X_L$ and $X_R$. I mean,

$$X_L \leq x \qquad \text{and} \qquad x \leq X_R. \tag{T2}$$

B. That shouldn't be hard to prove. What does $x_L \not\leq x$ say?

A. Either there is a number $x_{LL}$ in $X_{LL}$, with $x_{LL} \geq x$, or else there is a number $x_R$ in $X_R$ with $x_L \geq x_R$. But the second case can't happen, by rule (1).

B. I *knew* we were going to use rule (1) sooner or later. But what can we do with $x_{LL}$? I don't like double subscripts.

A. Well, $x_{LL}$ is an element of the left set of $x_L$. Since $x_L$ was created earlier than $x$, we can at least assume that $x_{LL} \leq x_L$, by induction.

B. Lead on.

A. Let's see, $x_{LL} \leq x_L$ says that $x_{LLL} \not\geq x_L$ and ...

B. (interrupting) I don't want to look at this — your subscripts are getting worse.

A. You're a big help.

B. Look, I *am* helping, I'm telling you to keep away from those hairy subscripts!

A：我觉得我们可以尝试一下之前发挥过作用的技术。如果 $x$ 和 $y$ 是这种意义下的坏数，那末，要么有某些 $x_L \geq y$，要么有 $x \geq$ 某些 $y_R$。

B：唔。假设 $y \leq x_L$ 好了，若 $x_L \leq x$，根据我们的传递律，有 $y \leq x$。但我们已经假定了 $y \not\leq x$，那就是说 $x_L \not\leq x$。再看另一种情况，$y_R \leq x$，同理可得 $y \not\leq y_R$。

A：嘿，这个结果可真是富有洞见呀！这下子，我们要做的工作就变成了这样：为了证明某个命题不成立，我们就要证明某个我很久以来都疑心为真的命题。任何数 $x$ 都必须位于它的左集和右集的所有元素之间。即，

$$X_L \leq x \qquad 且 \qquad x \leq X_R. \tag{T2}$$

B：那个应该不难证明吧。$x_L \not\leq x$ 实质上是在说什么呢？

A：或在 $X_{LL}$ 中有某个数 $x_{LL}$ 满足 $x_{LL} \geq x$，亦或在 $X_R$ 中有某个数 $x_R$ 满足 $x_L \geq x_R$。但是根据规则 (1)，后一种情况是不可能出现的。

B：我就知道早晚还是会用到规则 (1) 的。但是我们能对 $x_{LL}$ 做点儿加工么？我讨厌双重下标。

A：那个嘛，$x_{LL}$ 是 $x_L$ 的左集中的一个元素。由于 $x_L$ 比 $x$ 先创造出来，我们至少可以假设 $x_{LL} \leq x_L$，运用归纳法。

B：你说下去。

A：我们继续，$x_{LL} \leq x_L$ 表明 $x_{LLL} \not\geq x_L$，还有……

B：（打断）我可不想看到事情变成这样——你的下标变得更不像话了。

A：你可是帮了大忙了。

B：对，我是在帮忙，我是想让你远离这种纠缠不清的下标形式！

A. But I ... Okay, you're right, excuse me for going off on such a silly tangent. We have $x \leq x_{LL}$ and $x_{LL} \leq x_L$, so the transitive law tells us that $x \leq x_L$. This probably gets around the need for extra subscripts.

B. Aha, that does it. We can't have $x \leq x_L$, because that would mean $X_L \not\geq x_L$, which is impossible since $x_L$ is one of the elements of $X_L$.

A. Good point, but how do you know that $x_L \leq x_L$.

B. What? You mean we've come this far and haven't even proved that a number is like itself? Incredible ... there must be an easy proof.

A. Maybe you can see it, but I don't think it's obvious. At any rate, let's try to prove

$$x \leq x. \tag{T3}$$

This means that $X_L \not\geq x$ and $x \not\geq X_R$.

B. It's curiously like (T2). But uh-oh, here we are in the same spot again, trying to show that $x \leq x_L$ is impossible.

A. This time it's all right, Bill. Your argument shows that $x \leq x_L$ implies $x_L \not\geq x_L$, which is impossible by induction.

B. Beautiful! That means (T3) is true, so everything falls into place. We've got the "$X_L \leq x$" half of (T2) proved — and the other half must follow by the same argument, interchanging left and right everywhere.

A. And like we said before, (T2) is enough to prove that all numbers are related; in other words

$$\text{if} \quad x \not\leq y, \quad \text{then} \quad y \leq x. \tag{T4}$$

A：但是我……好吧，你是对的，请原谅我在这丛乱象中纠缠了那么久。我们有 $x \le x_{LL}$ 和 $x_{LL} \le x_L$，根据传递律可以得出 $x \le x_L$。这样做，也许就可以绕过额外的下标了吧。

B：啊哈，结果出来了。$x \le x_L$ 是不成立的，因为这个不等式意味着 $X_L \not\geq x_L$，而后者是矛盾的，因为 $x_L$ 是 $X_L$ 中的一个元素呀。

A：说得好，但是你又如何知道 $x_L \le x_L$。

B：什么？你是说我们做了这么多工作，却还连一个数相似于其自身都还没证出来？真是难以置信……这个证明肯定易如反掌吧。

A：大概你已经想到证法了，但在我看来这并不是显而易见的。无论如何，我们来试证一下

$$x \le x. \tag{T3}$$

这就是说，$X_L \not\geq x$ 且 $x \not\geq X_R$。

B：这和 (T2) 惊人地相似啊。但是，啊呀，我们又绕回来了，还是要证明 $x \le x_L$ 不成立。

A：这回没问题了，Bill。你的论据显示 $x \le x_L$ 蕴涵了 $x_L \not\geq x_L$，而这一点可以由归纳法证明不成立。

B：漂亮！这就意味着 (T3) 成立，现在一切各就各位了。我们终于证明了 (T2) 中"$X_L \le x$"这一半，而另一半肯定遵循着相同的论据，只是在各处将左右对调即可。

A：然后就像我们先前所说的，(T2) 的成立已经足以证明所有的数都相关，亦即

$$若 \quad x \not\leq y, \quad 则 \quad y \le x. \tag{T4}$$

B. Right. Look, now we don't have to bother saying things so indirectly any more, since "$x \not\geq y$" is exactly the same as "$x$ is less than $y$."

A. I see, it's the same as "$x$ is less than or like $y$ but not like $y$." We can now write

$$x < y$$

in place of $x \not\geq y$, and the original rules (1) and (2) look much nicer. I wonder why Conway didn't define things that way? Maybe it's because a third rule would be needed to define what "less than" means, and he probably wanted to keep down the number of rules.

B. I wonder if it's possible to have two different numbers that are like each other. I mean, can we have both $x \leq y$ and $x \geq y$ when $X_L$ is not the same as $Y_L$?

A. Sure, we saw something like that before lunch. Don't you remember, we found that $0 \leq y$ and $y \leq 0$ when $y = (\{-1\}, \emptyset)$. And I think $(\{0, 1\}, \emptyset)$ will turn out to be like $(\{1\}, \emptyset)$.

B. You're right. When $x \leq y$ and $x \geq y$, I guess $x$ and $y$ are effectively equal for all practical purposes, because the transitive law tells us that $x \leq z$ if and only if $y \leq z$. They're interchangeable.

A. Another thing, we've also got two more transitive laws. I mean

$$\text{if} \quad x < y \quad \text{and} \quad y \leq z, \quad \text{then} \quad x < z; \qquad \text{(T5)}$$
$$\text{if} \quad x \leq y \quad \text{and} \quad y < z, \quad \text{then} \quad x < z. \qquad \text{(T6)}$$

B. Very nice—in fact, these both follow immediately from (T1), if we consider "$x < y$" equivalent to "$x \not\geq y$." There's no need to use (T2), (T3), or (T4) in the proofs of (T5) and (T6).

A. You know, when you look over everything we've proved, it's really very pretty. It's amazing that so much flows out of Conway's two rules.

B：对的。你瞧，现在我们用不着啰啰嗦嗦、吞吞吐吐地说话了，因为"$x \not\geq y$"和"$x$ 小于 $y$"是一模一样的意思。

A：我懂了，也就是说"$x$ 小于或相似于 $y$，但是又不相似于 $y$"。我们现在尽可以把"$x \not\geq y$"直接写作

$$x < y,$$

并且最原始的规则 (1) 和 (2) 看起来也顺眼多了。我在想，Conway 为什么不直接那样给出定义呢？也许因为这么一来就需要第三条规则来定义何为"小于"，而他想尽可能压低规则的数目。

B：我在想，是否有可能存在两个不同的数彼此相似。就是说 $x \leq y$ 和 $x \geq y$ 都成立，但 $X_L$ 不同于 $Y_L$ 呢？

A：肯定有的，我们在午饭前看到过这种现象的。你不记得了吗，我们曾发现当 $y = (\{-1\}, \emptyset)$ 时，有 $0 \leq y$ 且 $y \leq 0$，并且我想 $(\{0,1\}, \emptyset)$ 和 $(\{1\}, \emptyset)$ 也是这样的关系。

B：你说得对。当 $x \leq y$ 和 $x \geq y$ 都成立时，它们在实践上完全可以说是相等的，因为由传递律可以推出 $x \leq z$ 当且仅当 $y \leq z$，两者在这里能够互换的。

A：还有，我们能够得到另外两条传递律。我是说

| | | | | | | |
|---|---|---|---|---|---|---|
| 若 | $x < y$ | 且 | $y \leq z$, | 则 | $x < z$; | (T5) |
| 若 | $x \leq y$ | 且 | $y < z$, | 则 | $x < z$. | (T6) |

B：真不错——事实上，如果我们考虑到"$x < y$"等价于"$x \not\geq y$"，这两条就可以从 (T1) 直接推得。在证明 (T5) 和 (T6) 时，其实都完全用不着 (T2)、(T3) 或 (T4)。

A：你看，当回头来看我们已经证明的这一切时，真的会有一种美感。从 Conway 的两条规则出发竟然可以推出这么多结果，真是妙不可言。

B. Alice, I'm seeing a new side of you today. You really put to rest the myth that women can't do mathematics.

A. Why, thank you, gallant knight!

B. I know it sounds crazy, but working on this creative stuff with you makes me feel like a stallion! You'd think so much brain-work would turn off any physical desires, but really — I haven't felt quite like this for a long time.

A. To tell the truth, neither have I.

B. Look at that sunset, just like in the poster we bought once. And look at that water.

A. (running) Let's go!

B：Alice，今天我看到了你的另一面。你可真是打破了“女性不宜搞数学”的神话啊。

A：怎么了嘛，谢谢你哟，我的骑士小嘴儿真甜！

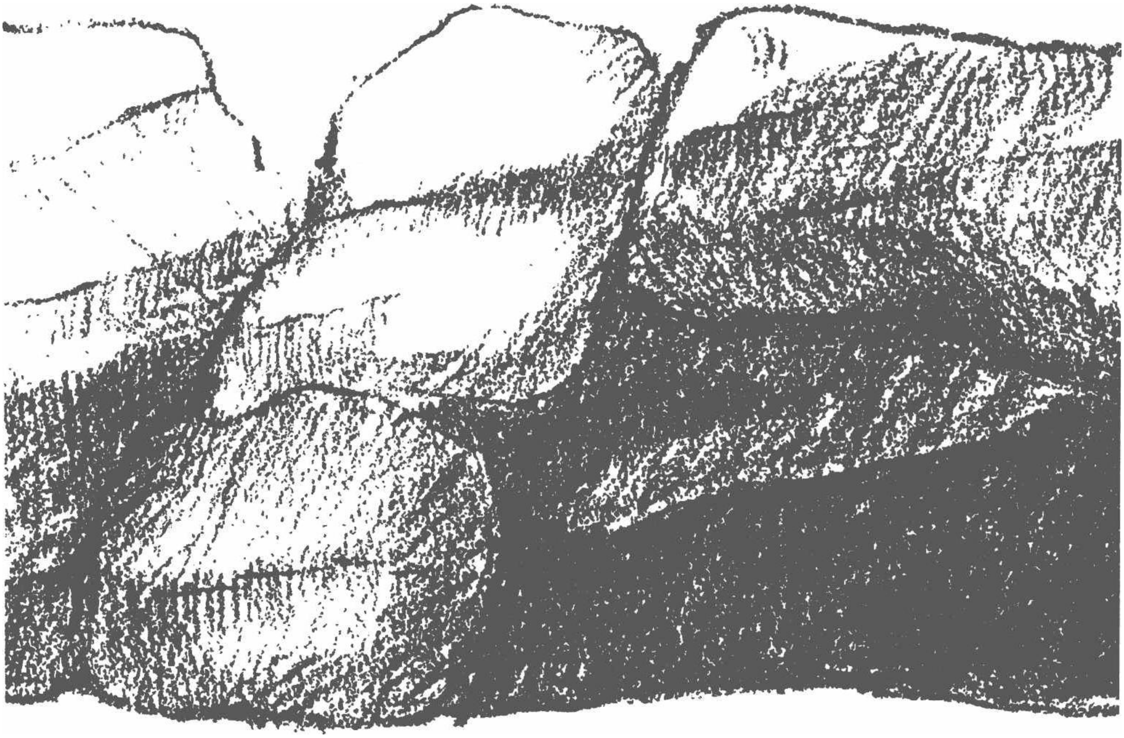
B：我知道听起来有点儿像是说胡话，但是和你一起做这样富有创意的工作让我感觉像匹牡马一样激情澎湃！你可能会觉得大量的脑力劳动会抑制掉任何物质欲望，但是，说实在的，我已经好长时间没体会到像今天这样酣畅淋漓的感觉了。

A：实话实说，我也一样很久没有过这种感觉了。

B：你看那夕阳，就跟我们有一回买的海报上画的一样。再看那海水。

A：（跑开）我们去游泳吧！

# 6 THE THIRD DAY



B. Boy, I never slept so well.

A. Me too. It's so great to wake up and be really awake, not just "coffee-awake."

B. Where were we yesterday, before we lost our heads and forgot all about mathematics?

A. (smiling) I think we had just proved that Conway's numbers behave like all little numbers should; they can be arranged in a

# 6　　第三日

B：人生啊，我从来没睡得这么好过。

A：我也是。这种起床以后很清醒、很舒服，而不是那种"被咖啡冲醒"的感觉真的好棒。

B：昨天我们进行到什么地方了，就是在我们犯了困并把所有数学的事儿丢到脑后之前？

A：（微笑）我觉得我们应该已经证明了 Conway 的数和所有较小的数有着相同的表现，它们可以

line, from smallest to largest, with every number being greater than those to its left and less than all those on its right.

B. Did we really prove that?

A. Yes. Anyway the unlike numbers keep in line, because of (T4). Every new number created must fall into place among the others.

B. Now it should be pretty easy for us to figure out what happened on the Third Day; those 20 × 20 calculations must be reduced 'way down. Our theorems (T2) and (T3) show that

⟨:▬⟩ < ▬ < ⟨▬:●⟩ < ● < ⟨●:❙⟩ < ❙ < ⟨❙:⟩

so seven of the numbers are placed already and it's just a matter of fitting the others in.

You know, now that it's getting easier, this is much more fun than a crossword puzzle.

A. We also know, for example, that

⟨▬:❙⟩

lies somewhere between ▬ and ❙. Let's check it against the middle element, zero.

B. Hmm, it's both ≤ and ≥ 0, so it must be *like* 0, according to rule (2). As I said yesterday, it's effectively equal to 0, so we might as well forget it. That's eight down and twelve to go.

A. Let's try to get rid of those ten cases where $X_L$ or $X_R$ have more than one element, like I tried to do yesterday morning. I had an idea during the night, which might work. Suppose $x = (X_L, X_R)$ is a number, and we take any other sets of numbers $Y_L$ and $Y_R$, where

$$Y_L < x < Y_R.$$

从小到大排成一列，每个数比位于它左边的数大，而比位于它右边的数小。

B：我们真的已经证明这件事了吗？

A：是的。根据 (T4)，在任何情况下，不同的数都可以排入队列。每一个新创造出来的数都肯定可以找到相对于其他数的位置。

B：现在，应该可以很容易推断出在第三日发生了什么。那 $20 \times 20$ 种运算必然可以省去大半，我们的定理 (T2) 和 (T3) 指出

⟨:▬⟩ < ▬ < ⟨▬:●⟩ < ● < ⟨●:▮⟩ < ▮ < ⟨▮:⟩

所以，已经七个数各就各位了，而问题就在于将其他的数放入合适的位置。

这么一来，现在要完成这件事已经容易多了，这可比纵横字谜要好玩。

A：我们知道，举例来说，

⟨▬:▮⟩

位于 ▬ 和 ▮ 之间的某个位置，我们就把它和中间的元素，也就是零，进行一下比较吧。

B：嗯，它既满足 $\leq 0$ 又满足 $\geq 0$，根据规则 (2)，它肯定相似于0。正如我昨天说的那样，它实际上就等于 0，那末我们也把它忘掉就可以了。现在有八个数已经安排妥当，还有十二个。

A：我现在试着把那些 $X_L$ 或 $X_R$ 包含多于一个元素的十种情况处理掉，就像我们昨天早上尝试过的那样。我昨晚产生了一个想法，也许会有用。假设 $x = (X_L; X_R)$ 是一个数，那么我们任意另取两个数集 $Y_L$ 或 $Y_R$ 满足

$$Y_L < x < Y_R$$

Then I think it's true that $x$ is like $z$, where

$$z = (X_L \cup Y_L, X_R \cup Y_R).$$

In other words, enlarging the sets $X_L$ and $X_R$, by adding numbers on the appropriate sides, doesn't really change $x$.

B. Let's see, that sounds plausible. At any rate, $z$ is a number, according to rule (1); it will be created sooner or later.

A. In order to show that $z \leq x$, we have to prove that

$$X_L \cup Y_L < x \qquad \text{and} \qquad z < X_R.$$

But that's easy, now, since we know that $X_L < x$, $Y_L < x$, and $z < X_R \cup Y_R$, by (T3).

B. And the same argument, interchanging left and right, shows that $x \leq z$. You're right, it's true:

$$\begin{aligned} &\text{if} \quad Y_L < x < Y_R, \\ &\text{then} \quad x \equiv (X_L \cup Y_L, X_R \cup Y_R). \end{aligned} \qquad \text{(T7)}$$

(I'm going to write "$x \equiv z$," meaning "$x$ is like $z$"; I mean $x \leq z$ and $z \leq x$.)

A. That proves just what we want. For example,

⟨—●:|⟩ ≡ ⟨●:|⟩,  ⟨:—●⟩ ≡ ⟨:—⟩

and so on.

B. So we're left with only two cases: ⟨—:⟩ and ⟨:|⟩.

A. Actually, (T7) applies to both of them, too, with $x = 0$!

B. Cle-*ver*. So the Third Day is now completely analyzed; only those seven numbers we listed before are essentially different.

那么,我认为在这种情况下有 $x$ 相似于 $z$,如果有

$$z = (X_L \cup Y_L, X_R \cup Y_R)$$

换句话说,采用将数字加到适当一侧的手法,对 $X_L$ 和 $X_R$ 进行扩张,其实改变不了 $x$。

B:我来瞧瞧,这听起来似是而非。无论如何,根据规则 (1),$z$ 的确是一个数,所以它迟早会被创造出来的。

A:为了得到 $z \le x$,我们必须证明

$$X_L \cup Y_L < x \qquad 且 \qquad z < X_R.$$

这个证明现在变得很容易了,因为根据 (T3),就能得出 $X_L < x$、$Y_L < x$ 以及 $z < X_R \cup Y_R$。

B:同理,左右交换,可以证得 $x \le z$。你说得对,下面这个定理成立:

$$\begin{aligned} &若 \quad Y_L < x < Y_R, \\ &则 \quad x \equiv (X_L \cup Y_L, X_R \cup Y_R). \end{aligned} \tag{T7}$$

(我打算采用“$x \equiv z$”的写法来表示“$x$ 相似于 $z$”,我是指 $x \le z$ 且 $z \le x$。)

A:这恰好证明了我们想要得到的。比如,

⟨—●:|⟩ ≡ ⟨●:|⟩,  ⟨:—●⟩ ≡ ⟨:—⟩

依此类推。

B:这么一来,我们只剩下两种情况了:⟨—:⟩ 和 ⟨:|⟩。

A:实际上,这两种情况也都可以适用 (T7),只要令 $x = 0$!

B:真——聪明。这么一来,现在第三日的情形已经完全分析好了。只有我们前面列出的那七个数才是本质上有所区别的。

A. I wonder if the same thing won't work for the following days, too. Suppose the different numbers at the end of $n$ days are

$$x_1 < x_2 < \cdots < x_m.$$

Then maybe the only new numbers created on the $(n+1)$st day will be

$$(\emptyset, \{x_1\}), \quad (\{x_1\}, \{x_2\}), \quad \ldots, \quad (\{x_{m-1}\}, \{x_m\}), \quad (\{x_m\}, \emptyset).$$

B. Alice, you're wonderful! If we prove this, it will solve infinitely many days in one swoop! You'll get ahead of the Creator himself.

A. But maybe we can't prove it.

B. Anyway let's try some special cases. Like, what if we had the number $(\{x_{i-1}\}, \{x_{i+1}\})$; it would have to be equal to one of the others.

A. Sure, it equals $x_i$, because of (T7). Look, each element of $X_{iL}$ is $\le x_{i-1}$, and each element of $X_{iR}$ is $\ge x_{i+1}$. Therefore, by (T7) we have

$$x_i \equiv (X_{iL} \cup \{x_{i-1}\}, X_{iR} \cup \{x_{i+1}\}).$$

And again by (T7),

$$(\{x_{i-1}\}, \{x_{i+1}\}) \equiv (\{x_{i-1}\} \cup X_{iL}, \{x_{i+1}\} \cup X_{iR}).$$

By the transitive law, $x_i \equiv (\{x_{i-1}\}, \{x_{i+1}\})$.

B. (shaking his head) Incredible, Holmes!

A. Elementary, my dear Watson. One simply uses deduction.

B. Your subscripts aren't very nice, but I'll ignore them this time. What would you do with the number $(\{x_{i-1}\}, \{x_{j+1}\})$ if $i < j$?

A：我想，会不会同样的规则在后续的日子中行不通了。假设经历了 $n$ 日时彼此相异的数有

$$x_1 < x_2 < \cdots < x_m.$$

那末，在第 $n+1$ 天，新创造的数仅有以下这些

$$(\emptyset, \{x_1\}),\ (\{x_1\}, \{x_2\}),\ \ldots,\ (\{x_{m-1}\}, \{x_m\}),\ (\{x_m\}, \emptyset).$$

B：Alice，你可真行啊！如果我们能证明这个假设，那就能将对于无数日的情况分析之功毕于一役了！你这个思想真是比“造数主”他老人家本人还要超前呢。

A：但也许我们无法证明呢？

B：无论如何，我们先尝试一些特殊情况好了。比如，对于数 $(\{x_{i-1}\}, \{x_{i+1}\})$，我们能得出什么结论呢？它应该会和其他数中的一个相等。

A：没错，根据 (T7)，它等于 $x_i$。你看，每个 $X_{iL}$ 中的元素皆满足 $\le x_{i-1}$，且每个 $X_{iR}$ 中的元素皆满足 $\ge x_{i+1}$。因此，根据 (T7) 可得

$$x_i \equiv (X_{iL} \cup \{x_{i-1}\},\ X_{iR} \cup \{x_{i+1}\}).$$

再次运用 (T7)，

$$(\{x_{i-1}\}, \{x_{i+1}\}) \equiv (\{x_{i-1}\} \cup X_{iL}, \{x_{i+1}\} \cup X_{iR}).$$

根据传递律，有 $x_i \equiv (\{x_{i-1}\}, \{x_{i+1}\})$。

B：（吃惊地摇头）难以置信，你简直是福尔摩斯啊！

A：小意思啦，亲爱的华生同志。这只不过是归纳法罢了。

B：你写的这些下标看起来不那么顺眼，不过这次我就不计较了。但是如果遇到像 $(\{x_{i-1}\}, \{x_{j+1}\})$，$i < j$ 这样的数，你又该怎么办呢？

A. (shrugging her shoulders) I was afraid you'd ask that. I don't know.

B. Your same argument would work beautifully if there was a number $x$ where each element of $X_L$ is $\leq x_{i-1}$ and each element of $X_R$ is $\geq x_{j+1}$.

A. Yes, you're right, I hadn't noticed that. But all those elements $x_i, x_{i+1}, \ldots, x_j$ in between might interfere.

B. I suppose so ... No, I've got it! Let $x$ be the one of $x_i, x_{i+1}, \ldots, x_j$ that was created *first*. Then $X_L$ and $X_R$ can't involve any of the others! So $(\{x_{i-1}\}, \{x_{j+1}\}) \equiv x$.

A. Allow me to give you a kiss for that.

. . . . . . . .

. . . . . . . .

B. (smiling) The problem isn't completely solved, yet; we have to consider numbers like $(\emptyset, \{x_{j+1}\})$ and $(\{x_{i-1}\}, \emptyset)$. But in the first case, we get the first-created number of $x_1, x_2, \ldots, x_j$. And in the second case it's the first-created number of $x_i, x_{i+1}, \ldots, x_m$.

A. What if the first-created number wasn't unique? I mean, what if more than one of the $x_i, \ldots, x_j$ were created on that earliest day?

B. Whoops ... No, it's okay, that can't happen, because the proof is still valid and it would show that the two numbers are both like each other, which is impossible.

A. Neato! You've solved the problem of all the days at once.

B. With your help. Let's see, on the fourth day there will be 8 new numbers, then on the fifth day there are 16 more, and so on.

A. Yes, after the $n$th day, exactly $2^n - 1$ numbers will have been created.

A：（耸肩）我就怕你问这个。我答不上来。

B：若是有数 $x$ 满足 $X_L$ 中的任意元素 $\le x_{i-1}$，且 $X_R$ 中的任意元素 $\ge x_{j+1}$，那么你前面的讨论在这里不是仍然可以完美地成立吗？

A：你说得没错，我还没想到这一点。但是那些位于中间的元素 $x_i, x_{i+1}, \ldots, x_j$ 不是会互相干扰吗？

B：说得也是……不对，我知道是怎么回事了！令 $x$ 作为 $x_i, x_{i+1}, \ldots, x_j$ 中最先被创造出来的数，那末 $X_L$ 和 $X_R$ 就无法包含任何其他元素了！所以，$(\{x_{i-1}\}, \{x_{j+1}\}) \equiv x$。

A：因为这个，请允许我奉上香吻一枚作为奖励。

…………

…………

B：（微笑）问题还没有彻底解决呢，我们必须要考虑像 $(\emptyset, \{x_{j+1}\})$ 和 $(\{x_{i-1}\}, \emptyset)$ 这样的数。但是在前一种情况中，该数是 $x_1, x_2, \ldots, x_j$ 中最先被创造的那个；而在后一种情况中，该数是 $x_i, x_{i+1}, \ldots, x_m$ 中最先被创造的那个。

A：如果最先被创造的数不唯一，该如何是好呢？我的意思是，如果在 $x_i, \ldots, x_j$ 中有不止一个数是在最早的那天被创造的话？

B：喔……不，没问题的，这种情况不会发生。因为上面的证明依然成立，这么一来就会出现存在两个数彼此相似的情形，而这是不可能的。

A：漂亮！你真是把所有日子的问题都一举解决了。

B：还不多亏了你给力。你看，第四日会有 8 个新数被创造出来，而第五日会多出 16 个数，依此类推。

A：对，第 $n$ 天过去后，就会有刚好 $2^n - 1$ 个新数被创造出来了。

B. You know, I don't think that guy Conway was so smart after all. I mean, he could have just given much simpler rules, with the same effect. There's no need to talk about sets of numbers, and all that nonsense; he simply would have to say that the new numbers are created between existing adjacent ones, or at the ends.

**C. Rubbish. Wait until you get to infinite sets.**

A. What was that? Did you hear something? It sounded like thunder.

B. I'm afraid we'll be getting into the monsoon season pretty soon.

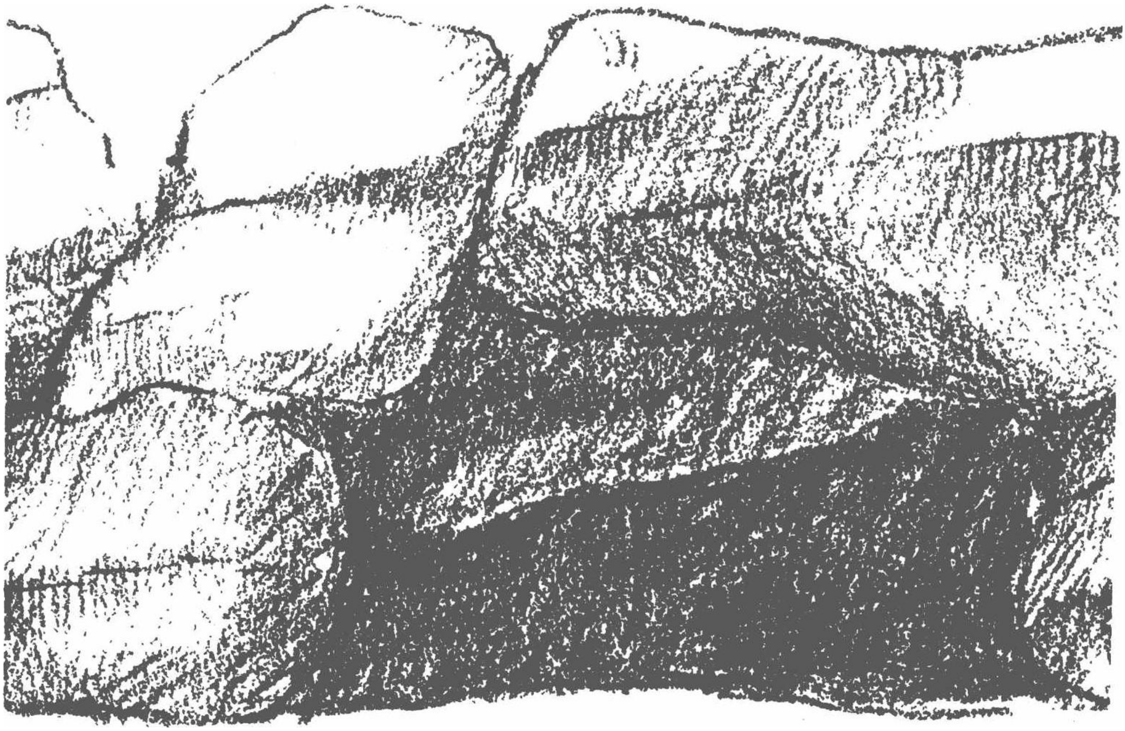
B：到了现在这个地步，我也并不觉得那个叫 Conway 的有多了不起。我的意思是，他完全可以给出简单得多的规则，达到同样的效果嘛。根本没必要讨论什么数集呀，还有这一整套把戏。他只要说，新数在相邻的两数之间或在边界处创建，不就行了。

**C：胡说八道。想想无限集合再说。**

A：那是什么？你听到什么声音了吗？好像打雷一样。

B：我们怕是快要进入季风时节了。

# 7 DISCOVERY

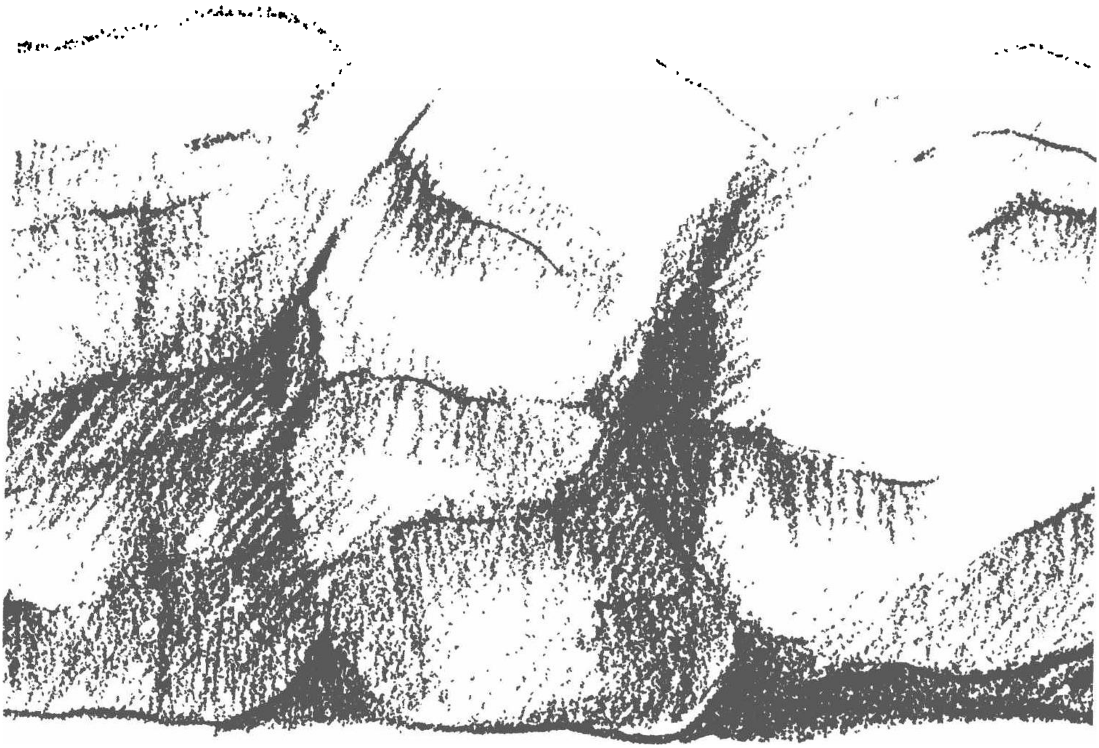


A. Well, we've solved everything on that rock, but I can't help feeling there's still a lot missing.

B. What do you mean?

A. I mean, like we know what happened on the third day; four numbers were created. But we don't know what Conway called them.

# 7　发　现



A：现在我们已经把那块岩石上的所有问题都解决了，但是我总觉得好像还缺了好多东西。

B：你说的是什么意思呢？

A：我的意思是说，我们现在知道第三日发生了什么，也就是一共有四个数被创造出来了。但我们并不知道 Conway 把它们叫做什么。

B. Well, one of the numbers was bigger than 1, so I suppose he called it "2." And another was between 0 and 1, so maybe he called it "$\frac{1}{2}$."

A. That's not the point. What really bothers me is, why are they *numbers*? I mean, in order to be numbers you have to be added, subtracted, and that sort of thing.

B. (frowning) I see. You think Conway gave some more rules, in the broken-off part of the rock, which made the numbers numerical. All we have is a bunch of objects ordered neatly in a line, but we haven't got anything to do with them.

A. I don't think I'm clairvoyant enough to guess what he did. If he did do something.

B. That means we're stuck — unless we can find the missing part of that rock. And I don't even remember where we found the first part.

A. Oh, I remember that; I was careful to note exactly where it was in case we ever wanted to go back.

B. What would I ever do without you? Come on, let's go!

A. Hey wait, don't you think we should have a little lunch first?

B. Right, I got so wrapped up in this I forgot all about food. Okay, let's grab a quick bite and then start digging.

. . . . . . . .

A. (digging) Oh, Bill, I'm afraid this isn't going to work. The dirt under the sand is so hard, we need special tools.

B. Yeah, just scraping away with this knife isn't getting us very far. Uh oh — here comes the rain, too. Should we dash back to camp?

B：嗯，一个数是比 1 大，所以我认为他叫它“2”。而另一个位于 0 和 1 之间，所以也许他叫它“$\frac{1}{2}$”。

A：我说的不是这个意思。我感觉真正苦恼的问题是，为何这些东西被叫做数呢？我是说，既然被叫做数，那它们就应该可以被用来做加法、减法，以及这类事情。

B：（皱起眉头）我明白了。你是觉得 Conway 应该会在那块岩石的断开部分给出了更多的规则，这就使那些数具备了数的性质。现在我们得到的只是有序地排成一列的对象，但我们并不能拿它们来做任何事。

A：我可不觉得我那么富有洞见，能够猜出他都做了哪些工作——如果他确实做了什么工作的话。

B：也就是说，我们卡在这里了——除非我们可以找到那块岩石被丢掉的那部分。可是我好像连现有的这一块在哪里找到的都忘记了。

A：啊，我倒是记得。我仔细地把那片地方做了标记，就等着什么时候我们可能会回去呢。

B：离了你，我肯定就一事无成！我们现在就去！

A：嗨，等等，我们是不是先把午饭解决一下？

B：对哦，我有点儿乐而忘食了。好，我们简单对付一下，然后开挖吧！

…………

A：（挖掘中）哦，Bill，恐怕不行。沙层下面的土方实在太硬，我们需要特殊的工具才成。

B：是啊，就用那种小刀片子，再划拉也不顶事儿。喔——要下雨了。我们打道回府吧。

A. Look, there's a cave over by that cliff. Let's wait out the storm in there. Hey, it's really pouring!

. . . . . . . .

B. Sure is dark in here. Ouch! I stubbed my toe on something. Of all the ...

A. Bill! You've found it! You stubbed your toe on the other part of the Conway Stone!

B. (wincing) Migosh, it looks like you're right. Talk about fate! But my toe isn't as pleased about it as the rest of me is.

A. Can you read it, Bill? Is it really the piece we want, or is it something else entirely?

B. It's too dark in here to see much. Help me drag it out in the rain, the water will wash the dust off and ...

Yup, I can make out the words "Conway" and "number," so it must be what we're looking for.

A. Oh, good, we'll have plenty to work on. We're saved!

B. The info we need is here all right. But I'm going back in the cave, it can't keep raining this hard for very long.

A. (following) Right, we're getting drenched.

. . . . . . . .

B. I wonder why this mathematics is so exciting now, when it was so dull in school. Do you remember old Professor Landau's lectures? I used to really hate that class: Theorem, proof, lemma, remark, theorem, proof, what a total drag.

A. Yes, I remember having a tough time staying awake. But look, wouldn't *our* beautiful discoveries be just about the same?

A：看呀，那里悬崖边上有个山洞。我们就在那里避避吧。嘿，雨下大了！

…………

B：这里可够暗的。哎哟！我的脚趾头撞到什么东西上了，真是……

A：Bill，你找到了！你的脚趾头是撞到 Conway 之岩的另一部分上了！

B：（疼得呲牙咧嘴）我的老天，看起来你是对的。天意啊！但我的脚趾头可不像身体的其他部分那么愉悦。

A：你可以读懂这个吗，Bill？这是我们在找的那块东西呢，还是完全不同的另一块岩石？

B：这里太暗，看不大清楚。来帮我一把，将这个东西拖出洞去，雨水会把尘土冲掉，还有……

啊，我可以辨认出来"Conway"和"数"这两个词了，所以这肯定是我们在找的东西。

A：太好了，我们会有很多事可做。我们得救了！

B：我们需要的信息全在这儿了。但我得钻回山洞里去，雨应该不会一直下这么大的。

A：（跟着进洞）不行了，我们都淋成落汤鸡了。

…………

B：我想不通为什么数学现在看起来这么让人兴奋有加，但在学校里就那么枯燥无味。你记得 Landau 老教授[1]上的课吗？我过去挺烦那门课的：定理、证明、引理、讨论、定理、证明……真是要命。

A：对啊，我记得我得拼命忍着才能不打瞌睡。但是看看，我们的伟大发现不也跟那些课的内容差不多吗？

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[1] 此处是暗指德国数学家 Edmund Landau (1877-1938)，他以严谨、繁复的证明和推理见长。

B. True. I've got this mad urge to get up before a class and present our results: Theorem, proof, lemma, remark. I'd make it so slick, nobody would be able to guess how we did it, and everyone would be *so* impressed.

A. Or bored.

B. Yes, there's that. I guess the excitement and the beauty comes in the discovery, not the hearing.

A. But it *is* beautiful. And I enjoyed hearing your discoveries almost as much as making my own. So what's the real difference?

B. I guess you're right, at that. I was able to really appreciate what *you* did, because I had already been struggling with the same problem myself.

A. It was dull before, because we weren't involved at all; we were just being told to absorb what somebody else did, and for all we knew there was nothing special about it.

B. From now on whenever I read a math book, I'm going to try to figure out by myself how everything was done, before looking at the solution. Even if I don't figure it out, I think I'll be able to see the beauty of a proof then.

A. And I think we should also try to guess what theorems are coming up; or at least, to figure out how and why anybody would try to prove such theorems in the first place. We should imagine ourselves in the discoverer's place. The creative part is really more interesting than the deductive part. Instead of concentrating just on finding good answers to questions, it's more important to learn how to find good questions!

B. You've got something there. I wish our teachers would give us problems like, "Find something interesting about $x$," instead of "Prove $x$."

A. Exactly. But teachers are so conservative, they'd be afraid of scaring off the "grind" type of students who obediently and

B：没错。我们每天起得比上课还早，忙着给出我们的结果：定理、证明、引理、讨论。我们做得这么顺手，没人能猜出我们怎么想到的，但是都会觉得那么地令人印象深刻。

A：或是不胜其烦。

B：对，你说到点子上了。我猜想兴奋和美感是来源于发现，而不是听别人说。

A：可是受教的确也很有美感啊。我在听你讲述你的发现，和自己亲自去发现同样享受。区别究竟在哪里呢？

B：我觉得你说得有道理，在这一点上。我也曾为你的成果而欣喜若狂。因为我自己也早已为同样的问题而奋战多时。

A：之所以我们以前会感觉厌烦，是由于我们完全没有参与进去。我们仅仅被告知要去吸收别人做出来的成果，所以我们对那些知识没有特别的感觉。

B：从今以后，我只要在读数学书的时候，都会在看解答之前，尝试自己去想想所有的东西是怎么得到的。就算自己得不到结果，也可以发现证明之美。

A：我想，我们还可以尝试猜想接踵而来的会有哪些定理，或至少可以猜想，为何人们一开始想要去证明这些定理。创新的部分真的是比那推导的部分有趣多了。比起为问题寻找更好的答案，学着去寻找更好的问题才是更重要的！

B：你确实说出了一些真知灼见。我希望我们的老师能给我们类似“寻找 $x$ 的一些有意思的性质”这样的问题，而不是“证明 $x$”。[2]

A：完全同意。但是老师们都太过保守，他们生怕把那些“当一天和尚撞一天钟”的那类学生给吓倒了。那些学生只知道顺从地、

[2] 有关“数学问题之位阶”的讨论，参见 Donald E. Knuth, *Concrete Mathematics*, 第 2 版，p72–73。

mechanically do all the homework. Besides, they wouldn't like the extra work of grading the answers to nondirected questions.

The traditional way is to put off all creative aspects until the last part of graduate school. For seventeen or more years, students are taught examsmanship; then suddenly after passing enough exams in graduate school they're told to do something original.

B. Right. I doubt if many of the really original students have stuck around that long.

A. Oh, I don't know, maybe they're original enough to find a way to enjoy the system. Like putting themselves into the subject, as we were saying. That would make the traditional college courses tolerable, maybe even fun.

B. You always were an optimist. I'm afraid you're painting too rosy a picture. But look, the rain has stopped. Let's lug this rock back to camp and see what it says.

机械地把作业做完。还有，他们不喜欢为获得没有固定导向的问题的答案而花费额外的时间。

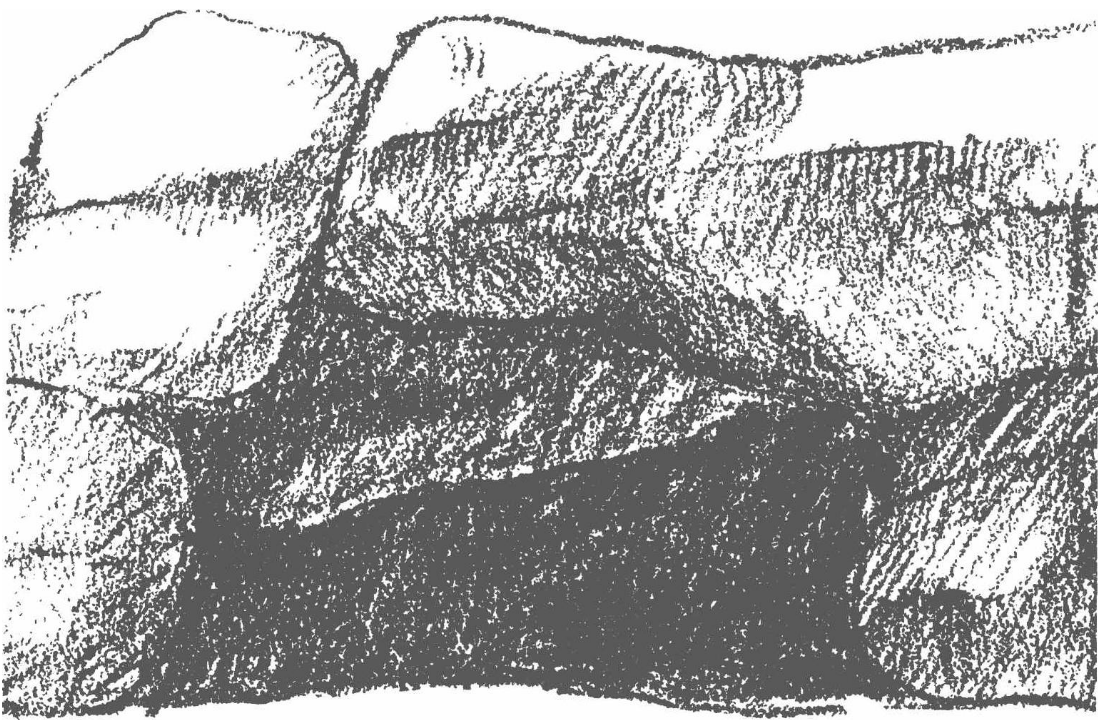
传统的做法是在直到学生毕业，都不会把创新的因素融入其中。也就是在十七年或更长的学习生涯中，学生们只被教导着去应试；尔后，在熬过了足够多的考试以后，他们突然被要求去做一些原创性的工作。

B：没错。我很怀疑那些真正有原创才能的学生能不能熬那么久。

A：哦，这我可说不好，也许他们的原创性足够强，以至于他们可以找到某种办法享受这个体制。比如，让自己投身于某些课题，就像我们刚才讨论的那样。这样做，也许可以使得传统的大学课程变得可以忍受，甚至充满乐趣了。

B：你总是这么个乐观主义者啊。不过，我觉得你恐怕把整件事想象得太过美好了。瞧，雨已经停了。我们把这块石头扛回营地，看看上面都说了些什么吧。

# 8 ADDITION

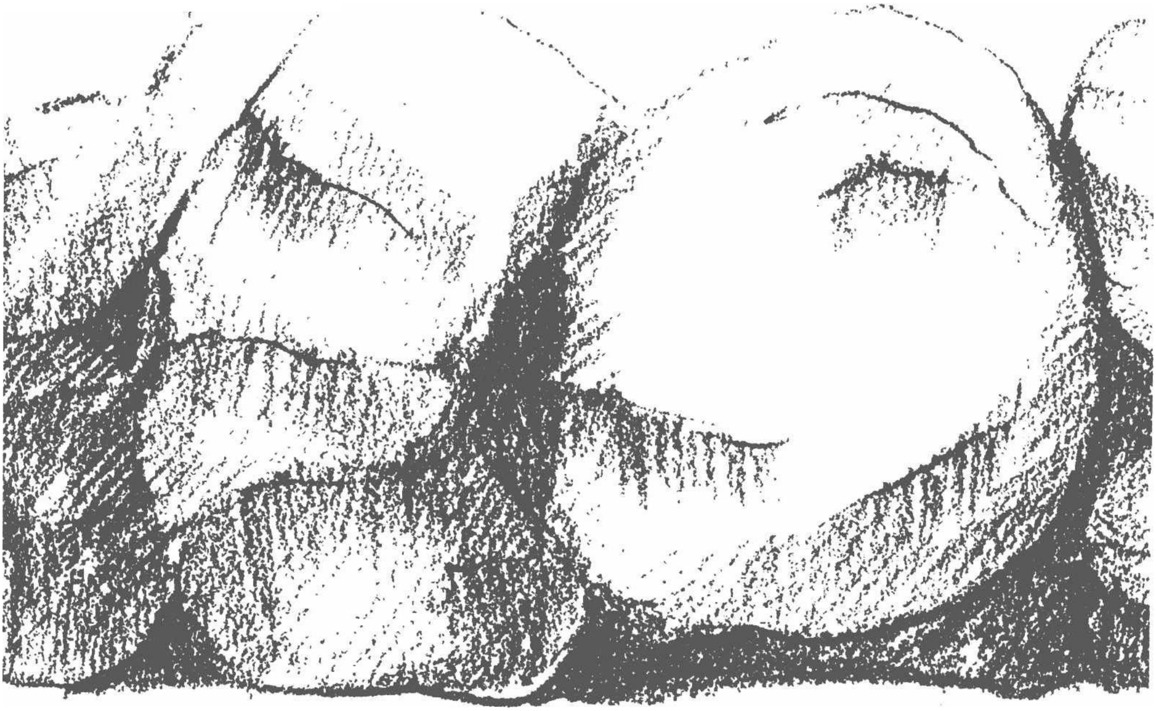


A. The two pieces fit pretty well, it looks like we've got almost the whole message. What does it say?

B. This part is a little harder to figure out — there are some obscure words — but I think it goes like this:

> **... day. And Conway said, "Let the numbers be added to each other in this wise: The left set of the sum of two numbers shall be the sums of all left parts of each number**

# 8　　加　法



A：两块岩石吻合得很好，看起来我们似乎已经掌握了几乎全部信息了。上面都说了些什么？

B：这个部分就有点儿令人费解——里面有些晦涩的单词——不过我想大体上应该是写了这些内容：

**……日。Conway 曰，“令数依此道相加：两数之和，其左集为每数之左部与另一数之和，**

with the other; and in like manner the right set shall be from the right parts, each according to its kind." Conway proved that every number plus zero is unchanged, and he saw that addition was good. And the evening and the morning were the third day.

And Conway said, "Let the negative of a number have as its sets the negatives of the number's opposite sets; and let subtraction be addition of the negative." And it was so. Conway proved that subtraction was the inverse of addition, and this was very good. And the evening and the morning were the fourth day.

And Conway said to the numbers, "Be fruitful and multiply. Let part of one number be multiplied by another and added to the product of the first number by part of the other, and let the product of the parts be subtracted. This shall be done in all possible ways, yielding a number in the left set of the product when the parts are of the same kind, but in the right set when they are of opposite kinds." Conway proved that every number times one is unchanged. And the evening and the morning were the fifth day.

And behold! When the numbers had been created for infinitely many days, the universe itself appeared. And the evening and the morning were ℵ day.

And Conway looked over all the rules he had made for numbers, and saw that they were very, very good. And he commanded them to be for signs, and series, and quotients, and roots.

Then there sprang up an infinite number less than infinity. And infinities of days brought forth multiple orders of infinities.

同理右集成于右部，依其类分。”Conway 证得，万数加零，其和不变，Conway 视之甚妙。夜去昼来，是为第三日。

Conway 又曰，“令负数之集为原数另集之负数所成，并令减去一数等于加其负数。”是为负数与减法。Conway 证得，减法为加法之逆，此甚妙。夜去昼来，是为第四日。

尔后，Conway 语诸数：“多产以乘。以一数之一部与另一数相乘之积，加该数与另一数之一部相乘之积，差以该两部自身相乘之积。依此做去，穷尽可能，终得积数：其左集为同部运算之所得，其右集为异部运算之所得也。”Conway 证明，万数乘一，其积不变。夜去昼来，是为第五日。

大哉！万数创生，无穷日逝，而宇宙现形。夜去昼来，是为第 ℵ 日。

尔后，Conway 检视创数诸道，连呼妙绝。后令其为正负、为级数、为商数、为方根。

后出一无穷数，不及玄极。逝日无穷，由是无穷亦高下有序也。

That's the whole bit.

A. What a weird ending. And what do you mean "aleph day"?

B. Well, aleph is a Hebrew letter and it's just standing there by itself, look: $\aleph$. It seems to mean infinity.

Let's face it, it's heavy stuff and it's not going to be easy to figure out what this means.

A. Can you write it all down while I fix supper? It's too much for me to keep in my head, and I can't read it.

B. Okay — that'll help me get it clearer in my own mind too.

. . . . . . . . .

A. It's curious that the four numbers created on the third day aren't mentioned. I still wonder what Conway called them.

B. Maybe if we try the rules for addition and subtraction we could figure out what the numbers are.

A. Yeah, *if* we can figure out those rules for addition and subtraction. Let's see if we can put the addition rule into symbolic form, in order to see what it means. ... I suppose "its own kind" must signify that left goes with left, and right with right. What do you think of this:

$$x + y = \big((X_L + y) \cup (Y_L + x),\ (X_R + y) \cup (Y_R + x)\big). \qquad (3)$$

B. Looks horrible. What does *your* rule mean?

A. To get the left set of $x + y$, you take all numbers of the form $x_L + y$, where $x_L$ is in $X_L$, and also all numbers $y_L + x$ where $y_L$ is in $Y_L$. The right set is from the right parts, "in like manner."

B. I see, a "left part" of $x$ is an element of $X_L$. Your symbolic definition certainly seems consistent with the prose one.

A. And it makes sense too, because each $x_L + y$ and $y_L + x$ ought to be less than $x + y$.

这就是全部内容了。

A：这个结尾还真是突兀啊。你这里的“阿列夫日”是什么意思啊？

B：怎么说呢，“阿列夫”是一个希伯来文字母，它就是很特立独行地放在那里，你瞧：ℵ。它貌似代表无穷大的意思。

面对它吧，这一大堆文字可是繁重的任务，要解释它的涵义可不是那么容易。

A：你能在我做晚餐的时候把这些东西写下来吗？要我一下子在脑子里记这么多东西，太多了，而且我又读不来那些字。

B：行——这也有助于让我把脑子里的信息整理整理。

…………

A：很有意思的一点是，这里只字未提在第三日创造的那四个数。我还是好奇 Conway 怎么称呼它们。

B：也许只要我们去尝试一下有关加法和减法的规则，就能推导出来这些数是什么了呢。

A：呵呵，*只要*我们能理解这些加法和减法的规则的话。试试看，我们是不是可以把加法的规则用符号形式表达出来，这样就可以明白它是什么意思了……我想，“依其类分”应该是“左与左、右与右”的意思吧。你看这个怎么样：

$$x + y = \big((X_L + y) \cup (Y_L + x), (X_R + y) \cup (Y_R + x)\big). \quad (3)$$

B：看起来好恐怖。那么可否解释一下*你的*规则？

A：为了获得 $x+y$ 的左集，就得包含所有形如 $x_L + y$ 的数，其中 $x_L$ 是 $X_L$ 中的元素；还包含形如 $y_L + x$ 的数，其中 $y_L$ 是 $Y_L$ 中的元素。而它的右集则取自右半部分，“同理可得”嘛。

B：我懂了，所谓 $x$ 的“左部”就是 $X_L$ 的任一元素。你给出的符号定义，当然看起来和叙述形式的定义是互洽的。

A：并且它的确也是有道理的，因为每个 $x_L + y$ 和 $y_L + x$ 都应该小于 $x + y$。

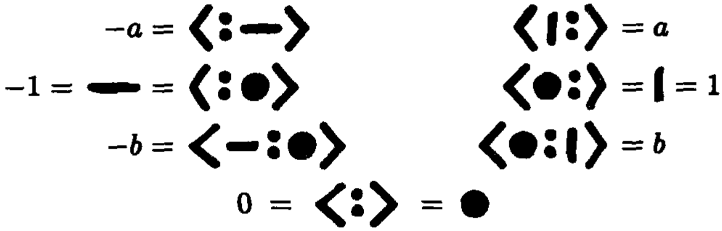
B. Okay, I'm willing to try it and see how it works. I see you've called it rule (3).

A. Now after the third day, we know that there are seven numbers, which we might call 0, 1, $-1$, $a$, $b$, $c$, and $d$.

B. No, I have an idea that we can use left-right symmetry and call them

$$-a < -1 < -b < 0 < b < 1 < a,$$

where




A. Brilliant! You must be right, because Conway's next rule is

$$-x \;=\; (-X_R, -X_L). \tag{4}$$

B. So it is! Okay — now we can start adding these numbers. Like, what's $1+1$, according to rule (3)?

A. You work on that, and I'll work on $1+a$.

B. Okay, I get $(\{0+1, 0+1\}, \emptyset)$. And $0+1$ is $(\{0+0\}, \emptyset)$; $0+0$ is $(\emptyset, \emptyset) = 0$. Everything fits together, making $1+1 = (\{1\}, \emptyset) = a$. Just as we thought, $a$ must be 2!

A. Congratulations on coming up with the world's longest proof that $1+1$ is 2.

B. Have you ever seen a shorter proof?

A. Not really. Look, your calculations help me too. I get $1 + 2 = (\{2\}, \emptyset)$, a number that isn't created until the fourth day.

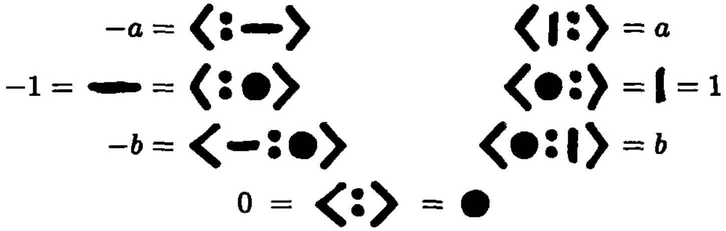
B：好吧，我真是跃跃欲试，看看它会如何运作。我看到，你把它称为规则 (3) 了。

A：现在，过了第三日时，我们已经知道一共有七个数，也许可以称它们为 0、1、−1、$a$、$b$、$c$，以及 $d$。

B：不，我有个主意，是不是可以采用一个左右对称的办法来表示

$$-a < -1 < -b < 0 < b < 1 < a,$$

其中




A：精辟！你肯定是对的，因为 Conway 的下一条规则是

$$-x = (-X_R, -X_L) \tag{4}$$

B：这就对啦！好 ——现在我们可以开始动手把这些数相加了，比如，根据规则 (3)，$1+1$ 是多少？

A：你做这个，我做 $1+a$。

B：好，我得到了 $(\{0+1, 0+1\}, \emptyset)$。而 $0+1$ 会得到 $(\{0+0\}, \emptyset)$；$0+0$ 会得到 $(\emptyset, \emptyset) = 0$。把所有的结果汇总起来，就得到 $1+1 = (\{1\}, \emptyset) = a$，正如我们猜想过的那样，$a$ 肯定就是 2！

A：祝贺你得到了世界上最长的对 $1+1$ 等于 2 的证明。[1]

B：你见过比这更短的证明吗？

A：还真没见过。不过，你的计算过程也启发了我。我得出了结论 $1+2 = (\{2\}, \emptyset)$，这是个到第四日才创造出来的数。

---

[1] 事实上，$1+1=2$ 的证明是数理逻辑学家的一大成果，他们用更为严格和晦涩的集合论公理来定义数和运算。如果有兴趣，可参见 A. N. Whitehead and B. Russell, *Principia Mathematica*，第 1 版，卷 1，p379，*54.43。

B. I suggest we call it "3."

A. Bravo. So rule (3) is working; let's check if $b$ is $\frac{1}{2}$ by calculating $b+b\ldots$

B. Hmm, that's odd, it comes out to $(\{b\},\{b+1\})$, which hasn't been created yet.

A. And $b+1$ is $(\{b,1\},\{2\})$, which is like $(\{1\},\{2\})$, which is created on the fourth day. So $b+b$ appears on the *fifth* day.

B. Don't tell me $b+b$ is going to be equal to *another* number we don't know the name of.

A. Are we stuck?

B. We worked out a theory that tells us how to calculate all numbers that are created, so we *should* be able to do this. Let's make a table for the first four days.

A. Oh, Bill, that's too much work.

B. No, it's a simple pattern really. Look:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Day 1 | | | | | | | | 0 | | | | | | | |
| Day 2 | | | | $-1$ | | | | | | | | 1 | | | |
| Day 3 | | $-2$ | | | | $-b$ | | | | $b$ | | | | 2 | |
| Day 4 | $-3$ | | $-(b+1)$ | | $-c$ | | $-d$ | | $d$ | | $c$ | | $b+1$ | | 3 |

A. Oh I see, so $b+b$ is $(\{b\},\{b+1\})$, which is formed from *non-adjacent* numbers ... and our theory says it is the *earliest-created* number between them.

B. (beaming) And that's 1, because 1 makes the scene before $c$.

A. So $b$ is $\frac{1}{2}$ after all, although its numerical value wasn't established until two days later. It's amazing what can be proved

B：我建议，我们就叫它“3”吧。

A：太好了。这么看来，规则 (3) 挺好用；现在来算一下 $b+b$，看看 $b$ 是不是 $\frac{1}{2}$……

B：唔，看起来有点儿怪，结果算出来是 $(\{b\},\{b+1\})$，一个还没有被创造出来的数。

A：并且 $b+1$ 的结果是 $(\{b,1\},\{2\})$，相似于 $(\{1\},\{2\})$，后者在第四日被创造。所以，$b+b$ 是在第五日被创造出来的。

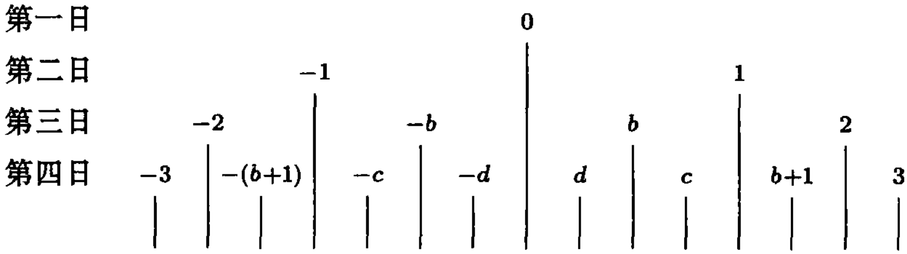
B：别告诉我 $b+b$ 的结果等于另一个我们不知道名字的数。

A：我们卡住了吗？

B：我们搞出来一个理论，用以计算所有创造出来的数，所以我们理应能够做到这一点。我们为前四日做一张表吧。

A：哦，Bill，这个工作量太大了吧。

B：不，这里面有个十分简单的模式在。你看：




A：哦，我看懂了，$b+b$ 算出来是 $(\{b\},\{b+1\})$，它由两个非相邻的数构成……而我们的理论指出，它就是这两个数之间最先被创造的数。

B：（眼前一亮）那它就是 1，因为 1 要比 $c$ 先问世。

A：所以 $b$ 到底还是 $\frac{1}{2}$，尽管它的数值是经过两日以后才确定下来的。从这寥寥几条规则出发能够证明的东西之多真是令人惊叹

from those few rules — they all hang together so tightly, it boggles the mind.

B. I'll bet $d$ is $\frac{1}{3}$ and $c$ is $\frac{2}{3}$.

A. But the sun is going down. Let's sleep on it, Bill; we've got lots of time and I'm really drained.

B. (muttering) $d + c = \ldots.$ Oh, all right. G'night.

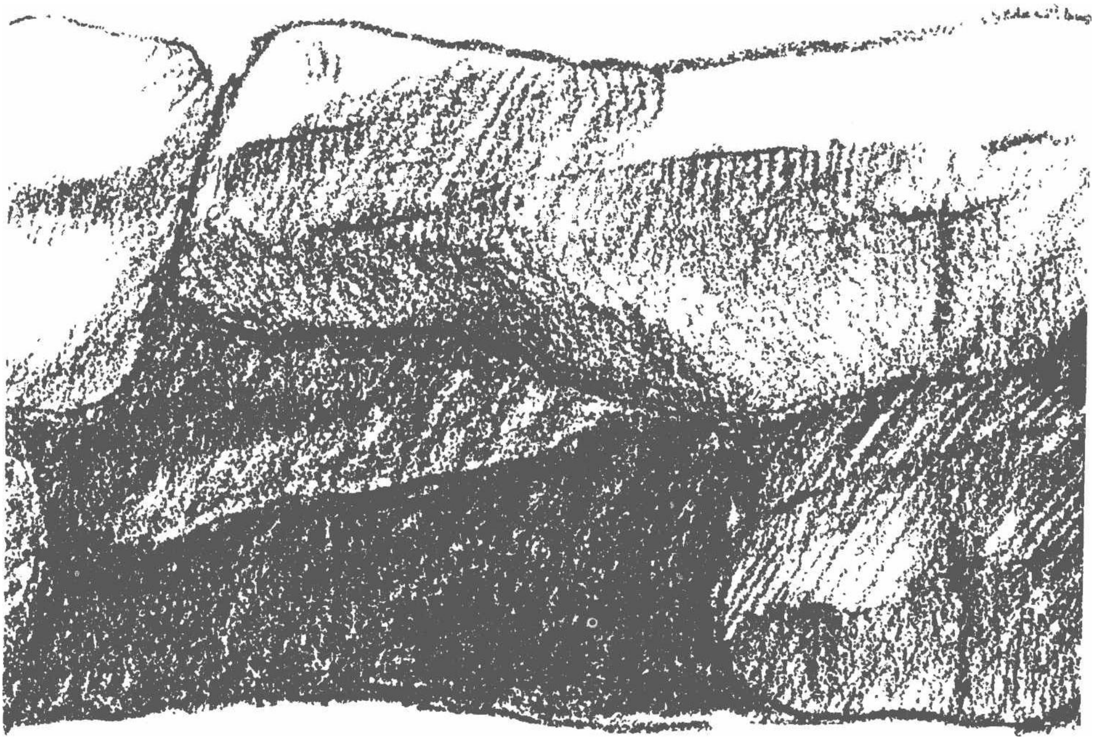
——它们彼此的联系是如此紧密，脑筋都有点儿转不过来了。

B：我敢打赌说，$d$ 是 $\frac{1}{3}$，而 $c$ 则是 $\frac{2}{3}$。

A：但是太阳已经落山啦。我们就此打点休息了吧，Bill；我们有的是时间，而且我真是连眼皮都抬不动了。

B：（咕哝着）$d + c = \ldots$，哦，好吧，晚安。

# 9 THE ANSWER

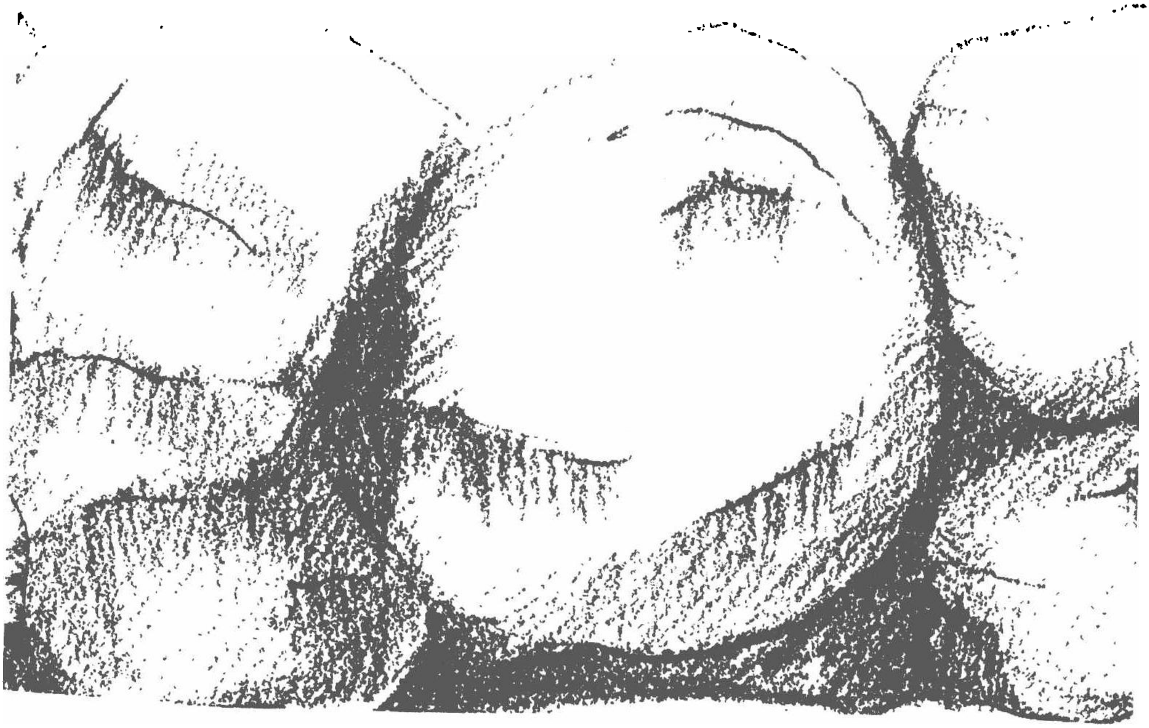


A. Are you awake already?

B. What a miserable night! I kept tossing and turning, and my mind was racing in circles. I dreamed I was proving things and making logical deductions, but when I woke up they were all foolishness.

A. Maybe this mathematics isn't good for us after all. We were so happy yesterday, but—

# 9 答 案



A：你醒啦？

B：真是心烦意乱的一夜啊！辗转反侧，不得入眠，并且我的思路在原地团团打转。我梦见自己一直在做证明，并且得到了一些符合逻辑的结论，但是醒来才发现其实只是在做梦。

A：也许这就是为什么数学对我们而言，毕竟还是无福消受呢。昨天我们还那么开心，现在可就……

B. (interrupting) Yeah, yesterday we were high on math, but today it's turning sour. I can't get it out of my system, we've *got* to get more results before I can rest. Where's that pencil?

A. Bill, you need some breakfast. There are some apricots and figs over there.

B. Okay, but I've gotta get right to work.

A. Actually I'm curious to see what happens too, but promise me one thing.

B. What?

A. We'll only work on addition and subtraction today; *not* multiplication. We won't even *look* at that other part of the tablet until later.

B. Agreed. I'm almost willing to postpone the multiplication indefinitely, since it looks awfully complicated.

A. (kissing him) Okay, now relax.

B. (stretching) You're so good to me, Alice.

A. That's better. Now I was thinking last night about how you solved the problem about all the numbers yesterday morning. I think it's an important principle that we ought to write down as a theorem. I mean:

> Given any number $y = (Y_L, Y_R)$, if $x$ is the first number created with the property that $Y_L < x$ and $x < Y_R$, then $x \equiv y$. (T8)

B. Hmm, I guess that *is* what we proved. Let's see if we can reconstruct the proof, in this new symbolism. As I recall we looked at the number $z = (X_L \cup Y_L, X_R \cup Y_R)$, for which we had $x \equiv z$ by (T7). On the other hand, no element $x_L$ of $X_L$ satisfies $Y_L < x_L$, since $x_L$ was created before $x$, and $x$ is supposed to be the oldest number with $Y_L < x$ and $x < Y_R$. Therefore

B：（打断话头）没错，昨天数学让我们尝到了甜头，今天却变成了苦涩的味道。但我已经陷入，无法全身而退，只能在休息之前做出更多的结果再说。我的铅笔呢？

A：Bill，你得弄点儿早饭垫一下。那里有些杏子和无花果什么的。

B：好吧好吧，但我想现在就动手干活啦。

A：其实我的心里也痒痒，很想看看我们能取得哪些结果，但你得答应我一件事。

B：什么事？

A：我们今天只研究加法和减法，不去碰乘法。我们对那块岩板上的其他文字看都不看一眼，等以后再说。

B：同意。我都恨不得永远不去碰乘法那部分，它看起来实在是复杂得太恐怖了。

A：（奉上香吻）好啦，现在放心了。

B：（伸懒腰）你对我太好啦，Alice。

A：你懂就好。我昨天晚上反思你昨天早上解决所有数字如何被创造的问题时，所采取的思路。我觉得这可是个重要的规律，得把它作为一个定理写下来。我是说：

给定任意数 $y = (Y_L, Y_R)$，若 $x$ 是创造出来的数中，
首个满足 $Y_L < x$ 及 $x < Y_R$ 的数，则有 $x \equiv y$。 (T8)

B：嗯，我想这的确是我们已经证得的。我来看看是不是可以把这个证明重构一下，采用新的符号体系。我回忆一下，在考察数 $z = (X_L \cup Y_L, X_R \cup Y_R)$ 时，根据 (T7) 我们得到了 $x \equiv z$。而从另一个角度看，$X_L$ 中不存在满足 $Y_L < x_L$ 的元素 $x_L$，这是因为 $x_L$ 比 $x$ 更早被创造，且 $x$ 被设定为满足 $Y_L < x$ 和 $x < Y_R$ 这两个条件的数中最早被创造的一个。所以，

each $x_L$ is $\leq$ some $y_L$, by (T4). Thus $X_L < y$, and similarly $y < X_R$. So $y \equiv z$ by (T7).

It's pretty easy to work out the proof now that we have all this ammunition to work with.

A. The nice thing about (T8) is that it makes the calculation we did last night much easier. Like when we were calculating $b + b = (\{b\}, \{b+1\})$, we could have seen immediately that 1 is the first number created between $\{b\}$ and $\{b+1\}$.

B. Hey, let me try that on $c + c$: It's the first number created between $b + c$ and $1 + c$. Well, it must be $b + 1$, I mean $1\frac{1}{2}$, so $c$ is $\frac{3}{4}$.

That's a surprise, I thought it would be $\frac{2}{3}$.

A. And $d$ is $\frac{1}{4}$.

B. Right.

A. I think the general pattern is becoming clear now: After four days the numbers $\geq 0$ are

$$0,\ \tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{3}{4},\ 1,\ \tfrac{3}{2},\ 2,\ 3$$

and after five days they will probably be—

B. (interrupting)

$$0,\ \tfrac{1}{8},\ \tfrac{1}{4},\ \tfrac{3}{8},\ \tfrac{1}{2},\ \tfrac{5}{8},\ \tfrac{3}{4},\ \tfrac{7}{8},\ 1,\ \tfrac{5}{4},\ \tfrac{3}{2},\ \tfrac{7}{4},\ 2,\ \tfrac{5}{2},\ 3,\ 4.$$

A. Exactly. Can you prove it?

B. ...

Yes, but not so easily as I thought. For example, to figure out the value of $f = (\{\frac{3}{2}\}, \{2\})$, which turned out to be $\frac{7}{4}$, I calculated $f + f$. This is the first number created between 3 and 4, and I had to "look ahead" to see that it was $\frac{7}{2}$. I'm convinced

根据 (T4)，$x_L$ 一定 $\leq$ 某些 $y_L$。综上所述，有 $X_L < y$。同理，有 $y < X_R$。根据 (T7)，可以推得 $y \equiv z$。

有了那么多理论武器以后，推得这个证明就易如反掌了。

A：(T8) 的优越之处在于，它使得我们进行昨天晚上那种计算的难度大大降低了。比如，我们在计算 $b+b=\bigl(\{b\},\{b+1\}\bigr)$ 时，就能够一眼看出 1 是 $b$ 和 $b+1$ 之间首个被创造的数。

B：嘿，让我在 $c+c$ 上算一下试试：它是 $b+c$ 和 $1+c$ 之间首个被创造的数。好吧，那个数肯定是 $b+1$，也就是 $1\frac{1}{2}$。所以，$c$ 是 $\frac{3}{4}$。

这可真是出乎意料，我原来以为它会是 $\frac{2}{3}$ 呢。

A：那么，$d$ 就是 $\frac{1}{4}$。

B：没错。

A：我觉得一个全局的模式现在变得非常清晰了：经过四天之后，$\geq 0$ 的数就有：

$$0,\ \tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{3}{4},\ 1,\ \tfrac{3}{2},\ 2,\ 3$$

那末，经过五天以后就会变成 ——

B：（打断）

$$0,\ \tfrac{1}{8},\ \tfrac{1}{4},\ \tfrac{3}{8},\ \tfrac{1}{2},\ \tfrac{5}{8},\ \tfrac{3}{4},\ \tfrac{7}{8},\ 1,\ \tfrac{5}{4},\ \tfrac{3}{2},\ \tfrac{7}{4},\ 2,\ \tfrac{5}{2},\ 3,\ 4.$$

A：一点不假。但是你能证明这个结论吗？

B：……

可以，但是这不像我乍一想的那样容易。举个例子来说，为了得到 $f=\bigl(\{\frac{3}{2}\},\{2\}\bigr)$ 的值，通过计算 $f+f$ 可知计算结果是 $\frac{7}{4}$。这是在 3 和 4 之间首个被创造出来的数，但我必须“提前发现”它是 $\frac{7}{2}$。我对于这个模式的正确性很有信心，

we have the right general pattern, but it would be nice to have a proof.

A. On the fourth day we calculated $\frac{3}{2}$ by knowing that it was $1 + \frac{1}{2}$, *not* by trying $\frac{3}{4} + \frac{3}{4}$. Maybe adding 1 will do the trick.

B. Let's see. ... According to the definition, rule (3),

$$1 + x = \big((1 + X_L) \cup \{x\},\ 1 + X_R\big),$$

assuming that $0 + x = x$. In fact, isn't it true that ... sure, for positive numbers we can always choose $X_L$ so that $1 + X_L$ has an element $\geq x$, so it simplifies to

$$1 + x = (1 + X_L, 1 + X_R)$$

in this case.

A. That's it, Bill! Look at the last eight numbers on the fifth day, they are just one greater than the eight numbers on the fourth day.

B. A perfect fit. Now all we have to do is prove the pattern for the numbers between 0 and 1 ... but that can always be done by looking at $x + x$, which will be less than 2!

A. Yes, now I'm sure we've got the right pattern.

B. What a load off my mind. I don't even feel the need to formalize the proof now; I ***know*** it's right.

A. I wonder if our rule for $1 + x$ isn't a special case of a more general rule. Like isn't

$$y + x = (y + X_L, y + X_R)?$$

That would be simpler than Conway's complicated rule.

但是如果能有一个证明就更好了。

A：在第四日时，我们计算出 $\frac{3}{2}$ 的途径是了解到它是 $1+\frac{1}{2}$ 结果，而并没有计算 $\frac{3}{2}+\frac{3}{2}$。也许通过加 1 就可以解决问题。

B：我们来试试看。……根据定义，规则 (3)，有

$$1+x=\big((1+X_L)\cup\{x\},1+X_R\big),$$

这要假定有 $0+x=x$。事实上，难道这个不成立吗……肯定成立的，对于所有的正数，我们总可以选择出这样的 $X_L$ 使得 $1+X_L$ 有一个元素 $\geq x$，所以这种情况下，上式可以简化为

$$1+x=(1+X_L,1+X_R)$$

A：这就对了，Bill！看看第五日的后面八个数，它们恰好比第四日的八个数分别大了一。

B：真是绝配。现在我们要做的全部工作就是证明 0 和 1 之间的数所具有的模式……而这些数就完全可以采用 $x+x$ 的办法算出来了，因为它们都小于 2！

A：是的，现在我可以确定我们已经找到了正确的模式了。

B：我心里一块大石头落地了！我觉得连写出正式的证明都不是很有必要了，我就知道这是对的。

A：我在想，我们有关 $1+x$ 的规则是否是另一个较普适规则的特殊情况，比如说，是否会有

$$y+x=(y+X_L,y+X_R)?$$

如果这个成立，那可比 Conway 的复杂规则要简单多了。

B. Sounds logical, since adding $y$ should "shift" things over by $y$ units. Whoops, no, take $x = 1$; that would say $y + 1$ is $(\{y\}, \emptyset)$, which fails when $y$ is $\frac{1}{2}$.

A. Sorry. In fact, your rule for $1 + x$ doesn't work when $x = 0$ either.

B. Right, I proved it only when $x$ is positive.

A. I think we ought to look at rule (3), the addition rule, more closely and see what can be proved in general from it. All we've got are *names* for the numbers. These names must be correct if Conway's numbers behave like actual numbers, but we don't know that Conway's rules are really the same. Besides, I think it's fun to derive a whole bunch of things from just a few basic rules.

B. Let's see. In the first place, addition is obviously what we might call commutative, I mean

$$x + y = y + x. \tag{T9}$$

A. True. Now let's prove what Conway claimed, that

$$x + 0 = x. \tag{T10}$$

B. The rule says that

$$x + 0 = (X_L + 0, X_R + 0).$$

So all we do is a "day of creation" induction argument, again; we can assume that $X_L + 0$ is the same as $X_L$, and $X_R + 0$ is $X_R$, since all those numbers were created before $x$. Q.E.D.

A. Haven't we proved that $x + 0 \equiv x$, not $= x$?

B. You're a nit-picker, you are. I'll change (T10) if you want me to, since it really won't make any difference. But doesn't the proof actually show that $x + 0$ is identically the same pair of sets as $x$?

B：听起来比较符合逻辑，因为加上 $y$ 就等于是所有的数“上移”$y$ 个单位。哦，不对了，取 $x=1$；这样的话，就会得到 $y+1$ 的结果为 $(\{y\},\emptyset)$，如果 $y=\frac{1}{2}$ 的话，结果就错了。

A：不好意思。其实，你那条有关 $1+x$ 的规则在 $x=0$ 时也不成立。

B：没错，我仅在 $x$ 为正的情况下证明了这个结论。

A：我觉得需要对规则 (3)，也就是加法规则，加以更仔细地审视，并想想从中能够推出怎么样的普适结论来。我们得到的只是各个数的名字。而如果 Conway 的数和实际的数一致，那么这些名字肯定是正确的，但是我们并不知道 Conway 的规则是不是真的和实际完全一样。另外一方面，我觉得能从寥寥几条规则出发就能推导出一大堆结论，倒是件很有趣的事。

B：我们来瞧瞧。首先一点，加法显然满足所谓的交换律，我的意思是

$$x+y=y+x. \tag{T9}$$

A：这个没错儿。现在我们来证明一下 Conway 所声称的那句话，即

$$x+0=x. \tag{T10}$$

B：加法规则表明

$$x+0=(X_L+0,X_R+0).$$

所以，我们要做的所有事就是把“数的创造日”归纳法再做一遍：我们可以假定 $X_L+0$ 和 $X_L$ 是等同的，$X_R+0$ 和 $X_R$ 是等同的，因为所有这些数都是在之前被创造出来的。证毕。

A：我们有没有证明 $x+0\equiv x$，而不仅仅是 $=x$?

B：你真会吹毛求疵啊，真是的。如果你一定要较真儿，我当然可以把 (T10) 改一下，但是我觉得改不改都没区别。这个证明实际上不是已经指出了 $x+0$ 和 $x$ 具有完全相同的集合对吗？

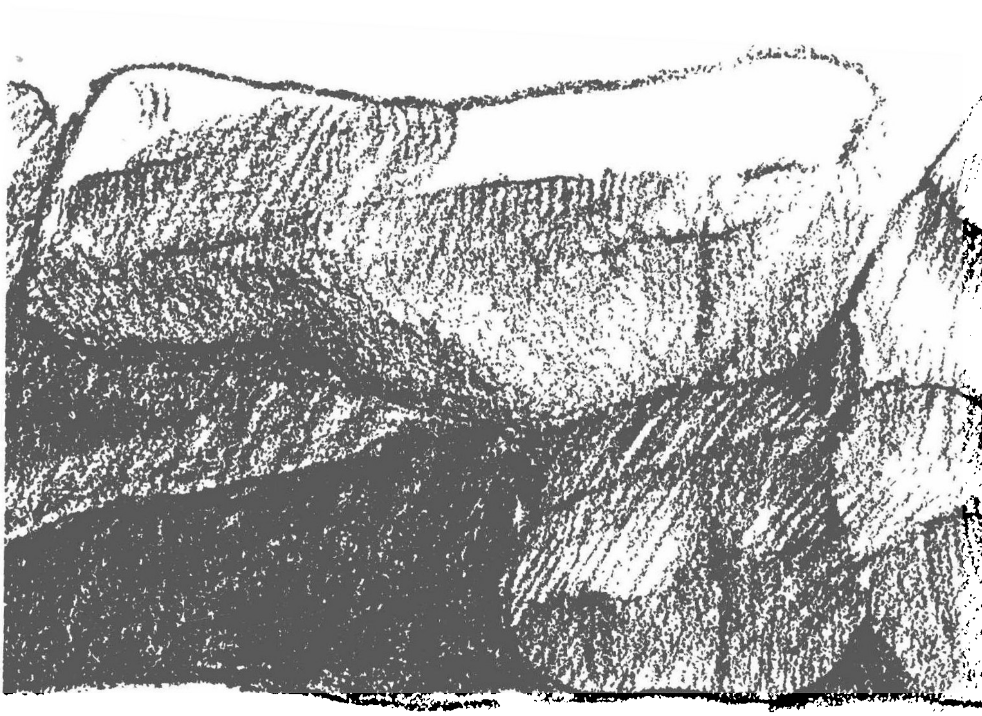
A. Excuse me again. You're right.

B. That's ten theorems. Should we try for more while we're hot?

A：不好意思，你是对的。

B：现在我们有十个定理了。我们是不是应该趁热打铁，得出更多的定理来呢？

# 10 THEOREMS

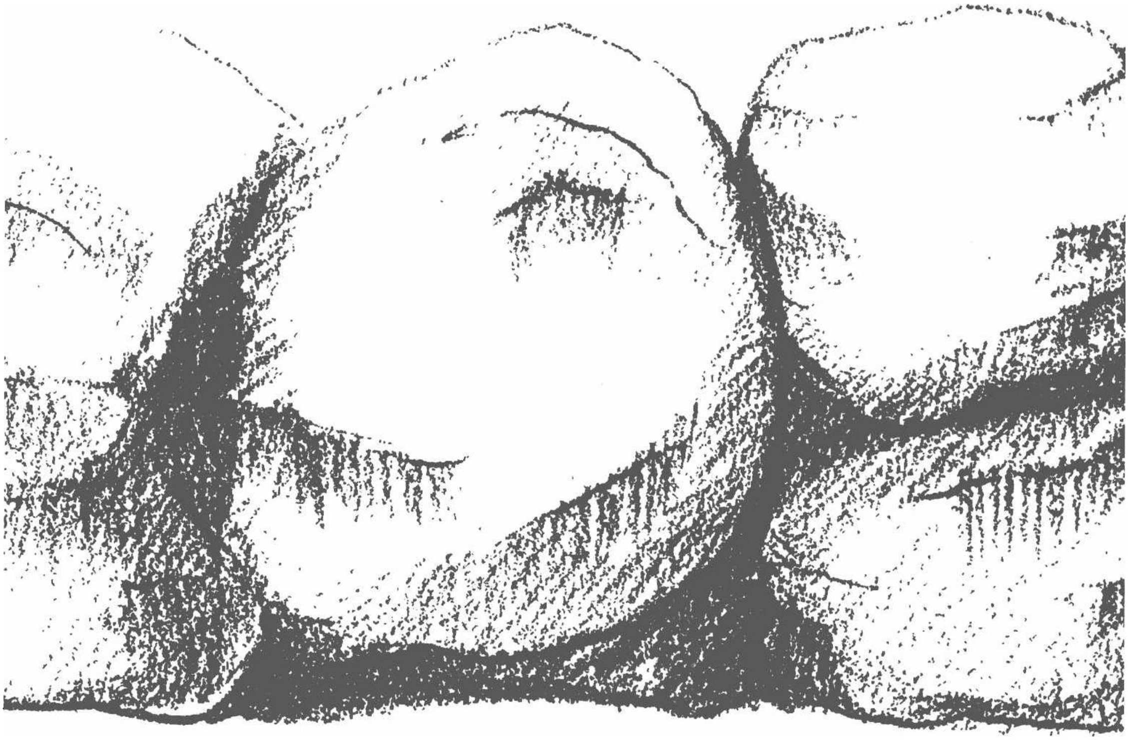


A. How about the associative law,

$$(x + y) + z = x + (y + z). \tag{T11}$$

B. Oh, I doubt if we'll need that; it didn't come up in the calculations. But I suppose it won't hurt to try it, since my math teachers always used to think it was such a great thing.

# 10 定理



A：先来看看结合律成立不成立吧，

$$(x+y)+z=x+(y+z). \tag{T11}$$

B：哦，我有点儿怀疑我们需不需要这条定理，因为它还从来没在计算中出现过。但是，把它证明出来也无伤大雅吧，好歹我的数学老师们仿佛都觉得这一条定理很重要似的。

One associative law, coming right up. Can you work out the definition?

A. $$(x+y)+z$$
$$= \big(((X_L+y)+z)\cup((Y_L+x)+z)\cup(Z_L+(x+y)),$$
$$((X_R+y)+z)\cup((Y_R+x)+z)\cup(Z_R+(x+y))\big)$$

$$x+(y+z)$$
$$= \big((X_L+(y+z))\cup((Y_L+z)+x)\cup((Z_L+y)+x),$$
$$(X_R+(y+z))\cup((Y_R+z)+x)\cup((Z_R+y)+x)\big).$$

B. You're really good at these hairy formulas. But how can such monstrous things be proved equal?

A. It's not hard, just using a day-sum argument on $(x,y,z)$ as we did before. See, $(X_L+y)+z = X_L+(y+z)$ because $(x_L,y,z)$ has a smaller day-sum than $(x,y,z)$, and we can induct on that. The same for the other five sets, using the commutative law in some cases.

B. Congratulations! Another Q.E.D., and another proof of "=" instead of "≡."

A. That ≡ worries me a little, Bill. We showed that we could substitute like elements for like elements, with respect to < and ≤, but don't we have to verify this also for addition? I mean,

$$\text{if} \quad x \equiv y, \quad \text{then} \quad x+z \equiv y+z. \tag{T12}$$

B. I suppose so; otherwise we wouldn't strictly be allowed to make the simplifications we've been making in our names for the numbers. As long as we're proving things, we might as well do it right.

A. In fact, we might as well prove a stronger statement,

$$\text{if} \quad x \le y, \quad \text{then} \quad x+z \le y+z, \tag{T13}$$

那就证证这条结合律,权当练手。你能把它的定义写出来吗?

A:
$$\begin{aligned}&(x+y)+z\\&\quad=\Big(((X_L+y)+z)\cup((Y_L+x)+z)\cup(Z_L+(x+y)),\\&\qquad((X_R+y)+z)\cup((Y_R+x)+z)\cup(Z_R+(x+y))\Big)\end{aligned}$$

$$\begin{aligned}&x+(y+z)\\&\quad=\Big((X_L+(y+z))\cup((Y_L+z)+x)\cup((Z_L+y)+x),\\&\qquad(X_R+(y+z))\cup((Y_R+z)+x)\cup((Z_R+y)+x)\Big).\end{aligned}$$

B:你对于处理这种乱糟糟的公式还真是挺在行啊。但是怎么才能证明这种鬼画符一样的等式呢?

A:其实不难,不过是再重复一遍我们对 $(x,y,z)$ 做过的那个有关创造日之和的讨论罢了。你看,$(X_L+y)+z=X_L+(y+z)$,因为 $(x_L,y,z)$ 比起 $(x,y,z)$ 来,创造日之和较小,然后我们就可以以此为基础进行归纳。同理,可以证明其他的五个集合对应地相等,其中可能要用到几次交换律。

B:万岁!又一条定理证毕,并且是一条关于"="关系而非"≡"关系的定理。

A:这个 ≡ 让我有点儿担心,Bill。我们已经得出,对 < 和 ≤ 的关系,我们可以把相似的元素相互替换。不过,难道我们不需要对于加法来验证这是否成立吗?我的意思是

$$\text{若}\quad x\equiv y,\quad\text{则}\quad x+z\equiv y+z. \tag{T12}$$

B:我觉得是这样的,否则,我们也不可能严格地以这样简化的方式来为我们的数命名了。既然我们现在就是在做证明的事儿,那索性就把它做对。

A:事实上,我们甚至还可以证明一条更强的定理,

$$\text{若}\quad x\le y,\quad\text{则}\quad x+z\le y+z, \tag{T13}$$

because this will immediately prove (T12).

B. I see, because $x \equiv y$ if and only if $x \leq y$ and $y \leq x$. Also (T13) looks like it will be useful. Shouldn't we also prove more, I mean

$$\text{if} \quad x \leq y \quad \text{and} \quad w \leq z,$$
$$\text{then} \quad x + w \leq y + z?$$

A. Oh, that follows from (T13), since $x + w \leq y + w = w + y \leq z + y = y + z$.

B. Okay, that's good, because (T13) is simpler. Well, you're the expert on formulas, what is (T13) equivalent to?

A. Given that $X_L < y$ and $x < Y_R$, we must prove that $X_L + z < y + z$, $Z_L + x < y + z$, $x + z < Y_R + z$, and $x + z < Z_R + y$.

B. Another day-sum induction, eh? Really, these are getting too easy.

A. Not quite so easy, this time. I'm afraid the induction will only give us $X_L + z \leq y + z$, and so on; it's conceivable that $x_L < y$ but $x_L + z \equiv y + z$.

B. Oh yeah. That's interesting. What we need is the converse,

$$\text{if} \quad x + z \leq y + z, \quad \text{then} \quad x \leq y. \tag{T14}$$

A. Brilliant! The converse is equivalent to this: Given that $X_L + z < y + z$, $Z_L + x < y + z$, $x + z < Y_R + z$, and $x + z < Z_R + y$, prove that $X_L < y$ and $x < Y_R$.

B. Hmm. The converse would go through by induction — except that we might have a case with, say, $x_L + z < y + z$ but $x_L \equiv y$. Such cases would be ruled out by (T13), but ...

A. But we need (T13) to prove (T14), and (T14) to prove (T13). And (T13) to prove (T12).

B. We're going around in circles again.

因为这一条定理可以立即证明 (T12)。

B：我明白了，因为当且仅当 $x \le y$ 且 $y \le x$ 时，$x \equiv y$。(T13) 看起来也会比较有用。我们不也可以更进一步的结论吗？我是说

$$\text{若} \quad x \le y \quad \text{且} \quad w \le z,$$
$$\text{则} \quad x + w \le y + z?$$

A：你说的这个可以直接从 (T13) 推出，因为 $x + w \le y + w = w + y \le z + y = y + z$。

B：好吧，那这样就很好，因为 (T13) 形式更简单。那么，你既然是公式达人，那你说说 (T13) 等价于什么？

A：已知 $X_L < y$ 且 $x < Y_R$，我们必须得证明 $X_L + z < y + z$、$Z_L + x < y + z$、$x + z < Y_R + z$，以及 $x + z < Z_R + y$。

B：又要来一次"创造日归纳法"，对不对？说实在的，现在这套把戏我们已经玩得很熟了。

A：这一回，可不太行得通了。恐怕运用归纳法只能证明出 $X_L + z < y + z$，依此类推；但是有可能虽然 $x_L < y$，却有 $x_L + z \equiv y + z$。

B：哦，对。这很有意思。我们实际上需要的是它的逆命题，

$$\text{若} \quad x + z \le y + z, \quad \text{则} \quad x \le y, \tag{T14}$$

A：聪明！这个逆命题等价于：已知 $X_L + z < y + z$、$Z_L + x < y + z$、$x + z < Y_R + z$，以及 $x + z < Z_R + y$，求证 $X_L < y$ 且 $x < Y_R$。

B：嗯。这个逆命题可以顺利地使用归纳法加以解决 ——除了有一种情况，$x_L + z < y + z$ 但是 $x_L \equiv y$。这种情况需要运用 (T13) 加以排除，但是……

A：但我们需要来 (T13) 证明 (T14)，并需要 (T14) 来证明 (T13)。而且，我们还需要 (T13) 来证明 (T12)。

B：我们又陷入循环论证的怪圈了。

A. Ah, but there's a way out, we'll prove them *both* together! We can prove the combined statement "(T13) and (T14)" by induction on the day-sum of $(x, y, z)$!

B. (glowing) Alice, you're a genius! An absolutely gorgeous, tantalizing genius!

A. Not so fast, we've still got work to do. We had better show that

$$x - x \equiv 0. \tag{T15}$$

B. What's that minus sign? We never wrote down Conway's rule for subtraction.

A. $$x - y = x + (-y). \tag{5}$$

B. I notice you put the $\equiv$ in (T15); okay, it's clear that $x + (-x)$ won't be identically equal to 0, I mean with empty left and right sets, unless $x$ is 0.

A. Rules (3), (4), and (5) say that (T15) is equivalent to this:

$$\big((X_L + (-x)) \cup ((-X_R) + x),\\ (X_R + (-x)) \cup ((-X_L) + x)\big) \equiv 0.$$

B. Uh oh, it looks hard. How do we show something $\equiv 0$ anyway? ... By (T8), $y \equiv 0$ if and only if $Y_L < 0$ and $0 < Y_R$, since 0 was the first created number of all.

A. The same statement also follows immediately from rule (2); I mean, $y \leq 0$ if and only if $Y_L < 0$, and $0 \leq y$ if and only if $0 < Y_R$. So now what we have to prove is

$$\begin{array}{llll} & x_L + (-x) < 0, & \text{and} & (-x_R) + x < 0, \\ \text{and} & x_R + (-x) > 0, & \text{and} & (-x_L) + x > 0, \end{array}$$

for all $x_L$ in $X_L$ and all $x_R$ in $X_R$.

A：啊哈，看起来有一条出路哩，我们可以一举两得地证明这两条定理！我们可以通过对 $(x, y, z)$ 运用创造日之和的归纳法来证明组合命题"(T13) 且 (T14)"。

B：（喜不自胜）Alice，你真是个天才！一个超级美丽又动人的天才！

A：慢着，还没大功告成呢。我们最好能证明

$$x - x \equiv 0. \tag{T15}$$

B：那个减号是什么意思？我们还从来没有把 Conway 有关减法的规则写下来过呢。

A：
$$x - y = x + (-y). \tag{5}$$

B：我发现你在 (T15) 中放入了一个 $\equiv$。好吧，显然 $x + (-x)$ 不可能与 0 完全等同，我是说，不可能左集和右集都是空集，除非 $x$ 是 0。

A：根据规则 (3)、(4) 和 (5)，(T15) 等价于这个式子：

$$\big((X_L + (-x)) \cup ((-X_R) + x),\ (X_R + (-x)) \cup ((-X_L) + x)\big) \equiv 0.$$

B：哦，这个看起来可有点儿头疼啊。我们到底怎么才能证明某个数 $\equiv 0$ 呢？……根据 (T8)，$y \equiv 0$ 当且仅当 $Y_L < 0$ 且 $0 < Y_R$，因为 0 是所有数中首个被创造的。

A：这些都可以从规则 (2) 立即推得，我的意思是，$y \le 0$ 当且仅当 $Y_L < 0$，而 $0 \le y$ 当且仅当 $0 < Y_R$。所以，现在我们需要证明的是对于 $X_L$ 中所有的 $x_L$ 和 $X_R$ 中的所有 $x_R$，有

$$\begin{array}{llll} & x_L + (-x) < 0, & 且 & (-x_R) + x < 0, \\ 且 & x_R + (-x) > 0, & 且 & (-x_L) + x > 0, \end{array}$$

B. Hmm. Aren't we allowed to assume that $x_L + (-x_L) \equiv 0$ and $x_R + (-x_R) \equiv 0$?

A. Yes, since we can be proving (T15) by induction.

B. Then I've got it! If $x_L + (-x)$ were $\geq 0$, then $(-X)_R + x_L$ would be $> 0$, by definition. But $(-X)_R$ is $-(X_L)$, which contains $-x_L$, and $(-x_L)+x_L$ is not $> 0$. Therefore $x_L+(-x)$ must be $< 0$, and the same technique works for the other cases too.

A. Bravo! That settles (T15).

B. What next?

A. How about this?

$$-(-x) = x. \tag{T16}$$

B. Sssss. That's trivial. Next?

A. All I can think of is Conway's theorem,

$$(x + y) - y \equiv x. \tag{T17}$$

B. What's that equivalent to?

A. It's a real mess. ... Can't we prove things without going back to the definitions each time?

B. Aha! Yes, it almost falls out by itself:

$$\begin{aligned}
(x + y) - y &= (x + y) + (-y) && \text{by (5)} \\
&= x + (y + (-y)) && \text{by (T11)} \\
&= x + (y - y) && \text{by (5)} \\
&\equiv x + 0 && \text{by (T12) and (T15)} \\
&= x && \text{by (T10).}
\end{aligned}$$

We've built up quite a pile of useful results — even the associative law has come in handy. Thanks for suggesting it against my better judgment.

B：唔。我们可不可以假定 $x_L + (-x_L) \equiv 0$ 且 $x_R + (-x_R) \equiv 0$?

A：可以吧，只要我们可以运用归纳法把 (T15) 证明出来。

B：那我就明白了！如果 $x_L + (-x)$ 是 $\geq 0$ 的，那么根据定义，$(-X)_R + x_L$ 就应该是 $> 0$ 的。但是 $(-X)_R$ 就是 $-(X_L)$，后者包含 $-x_L$，而 $(-x_L) + x_L$ 并非 $> 0$ 的。所以，$x_L + (-x)$ 肯定是 $< 0$ 的，运用同样的技术，另一种情况也可以得到证明。

A：太棒了，这么一来 (T15) 就得证了。

B：下一条定理是什么？

A：这一条怎么样？

$$-(-x) = x. \tag{T16}$$

B：切，这个不值一证嘛。下一条呢？

A：我能想到的就是 Conway 的定理，

$$(x+y) - y \equiv x. \tag{T17}$$

B：它等价于什么？

A：这真是一塌糊涂了……我们难道就不能不要每证一条定理都要回到原始定义去吗？

B：啊哈！可以的，这条定理几乎都不证自明了：

$$\begin{aligned}
(x+y) - y &= (x+y) + (-y) && \text{根据 (5)} \\
&= x + (y + (-y)) && \text{根据 (T11)} \\
&\equiv x + 0 && \text{根据 (T12) 和 (T15)} \\
&= x && \text{根据 (T10).}
\end{aligned}$$

我们已经建立了颇一整套实用的规则——即使是结合律也凑了个数帮上了忙。多亏你的建议加上了这一条，才避免让我自作聪明了一回。

A. Okay, we've probably exhausted the possibilities of addition, negation, and subtraction. There are some more things we could probably prove, like

$$-(x+y) = (-x) + (-y), \tag{T18}$$

$$\text{if} \quad x \le y, \quad \text{then} \quad -y \le -x, \tag{T19}$$

but I don't think they involve any new ideas; so there's little point in proving them unless we need 'em.

B. Nineteen theorems, from just a few primitive rules.

A. Now you must remember your promise: This afternoon we take a vacation from mathematics, without looking at the rest of the stone again. I don't want that horrible multiplication jazz to rob you of any more sleep.

B. We've done a good day's work, anyhow — all the problems are resolved. Look, the tide's just right again. Okay — the last one into the water has to cook supper!

A：好，我们很可能已经穷尽了加法、求相反数和减法的所有可能的定理了。也许还有更多我们可能可以证明的，比如

$$-(x+y)=(-x)+(-y) \tag{T18}$$

$$\text{若}\quad x\le y,\quad \text{则}\quad -y\le -x, \tag{T19}$$

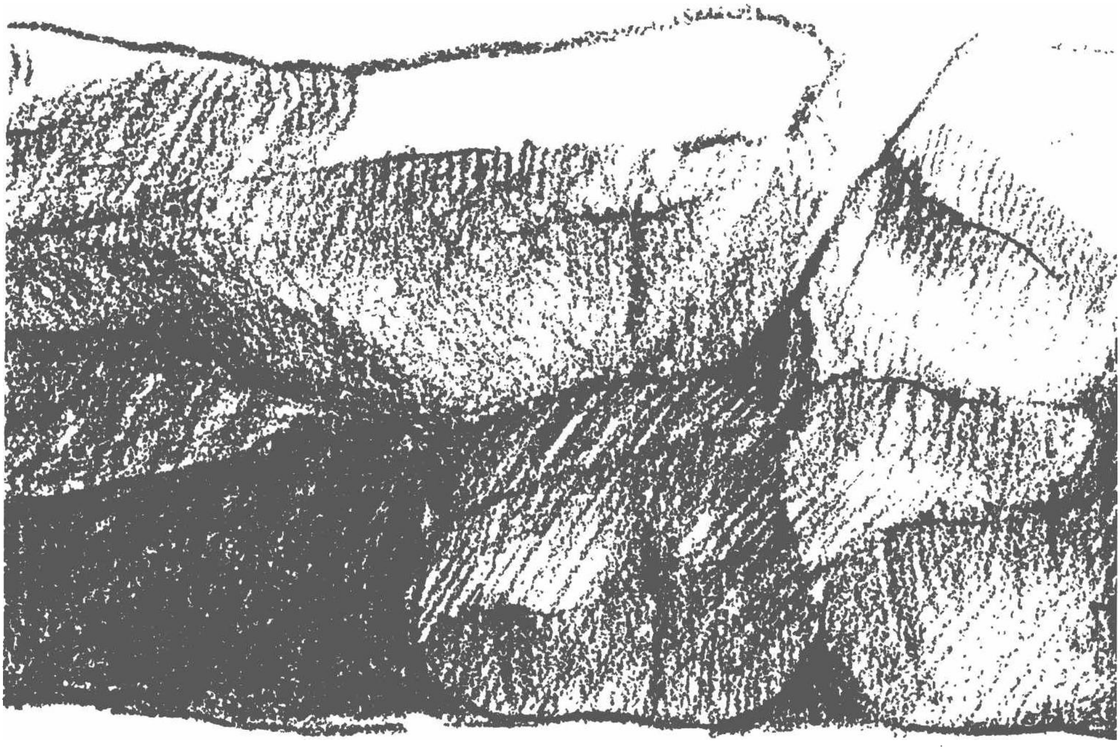
但我不觉得这些定理中包含任何新的思想，所以除非很有必要，给出证明就是没多大意义的事了。

B：十九条定理，仅仅从寥寥几条规则就可以导出来。

A：现在你得兑现你的承诺：今天下午我们彻底从数学中解放一个下午，并且不再朝岩石的其余部分多看一眼。我可不想让那段有关乘法的鬼画符再让你不得安眠，呵呵。

B：无论如何，我们今天的成果很不错——所有的问题一扫而光。你看，一切再次运行如仪。好了——晚下水的那个今天晚上做饭！

# 11 THE PROPOSAL



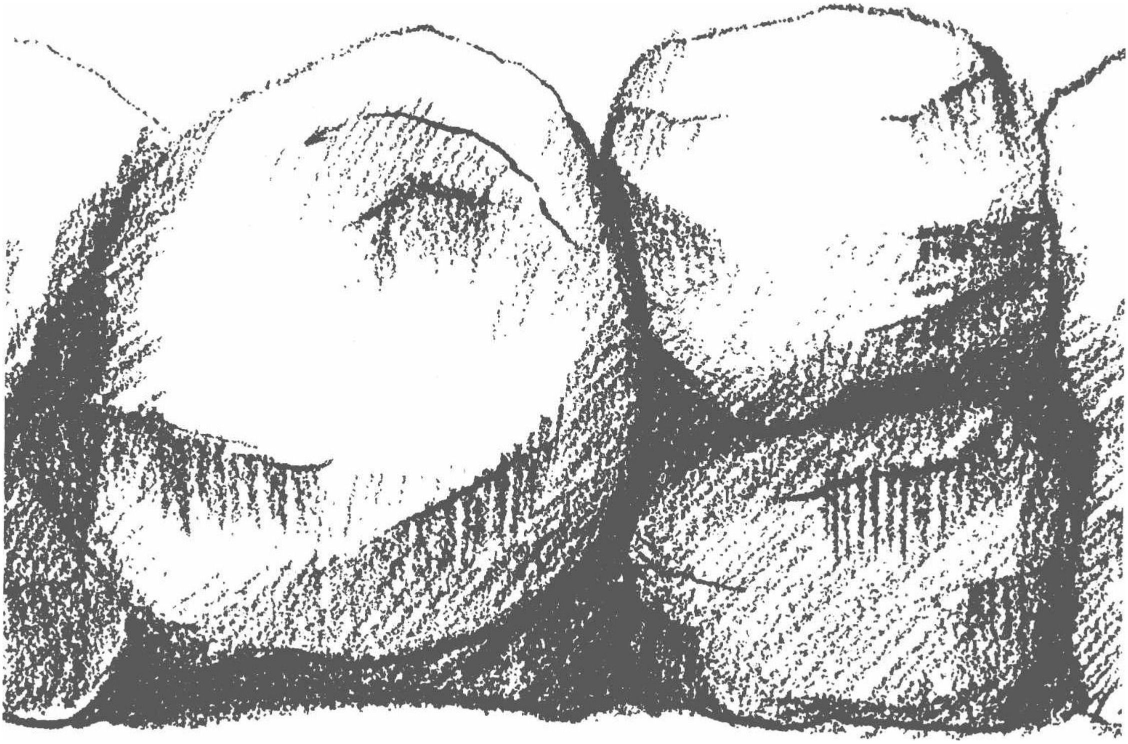
A. That sure was a good supper you cooked.

B. (lying down beside her) Mostly because of the fresh fish you caught.

What are you thinking about now?

A. (blushing) Well, actually I was wondering what would happen if I got pregnant.

# 11　缘　定



A：你做的晚饭真香啊。

B：（躺在她旁边）还不多亏你抓来的新鲜鱼儿。

你现在想什么呢？

A：（脸红）那个，其实我在想假如我有了孩子会是怎么样的。

B. You mean, here we are, near the Fertile Crescent, and ...?

A. Very funny. And after all our work to prove that $1 + 1 = 2$ we'll discover that $1 + 1 = 3$.

B. Okay, you win, no more jokes. But come to think of it, Conway's rules for numbers are like copulation, I mean the left set meeting the right set, ...

A. You've got just one thing—no, two things—on your mind. But seriously, what would we do if I really were pregnant?

B. Well, I've been thinking we'd better go back home pretty soon anyway; our money's running out, and the weather is going to get bad.

Actually, I really want to marry you in any case, whether you're pregnant or not. If you'll have me, of course.

A. That's just what I feel too. This trip has proved that we're ready for a permanent relationship.

I wonder ... When our children grow up, will we teach them our theory of numbers?

B. No, it would be more fun for them to discover it for themselves.

A. But people can't discover *everything* for themselves; there has to be some balance.

B. Well, isn't all learning really a process of self-discovery? Don't the best teachers help their students to think on their own?

A. In a way, yes. Whew, we're getting philosophical.

B. I still can't get over how great I feel when I'm doing this crazy mathematics; it really turns me on right now, but I used to hate it.

A. Yes, I've been high on it, too. I think it's a lot better than drugs; I mean, the brain can stimulate itself naturally.

B. And it was kind of an aphrodisiac, besides.

B：你不会是说，在这里，在这新月沃土[1]附近……吧？

A：那就有意思了。我们费了这么大的劲儿去证明 $1+1=2$，到头来却发现 $1+1=3$。

B：你可真行，不开玩笑了啊。不过回头想想，Conway 的这些有关数的规则还真有点儿像繁衍后代。你看，当左集遇上右集……

A：我看你脑子现在只有一个东西 ——哦，就算是两个东西吧。但是，说正经的，如果我真有了孩子的话，该怎么办呀？

B：那样的话，我们无论如何就得快点儿回家才行了。我们带出来的钱已经所剩无几，而且天气眼看着也要变糟了。

说真的，无论如何我都想和你永结同好，不管你有没有孩子。当然前提是你也想要嫁我。

A：我当然也有同样的感觉。这次旅行已经证明了我们已经为永久的关系做好了准备。

我在想……如果我们的孩子长大以后，我们会把这套有关数的理论教给他们吗？

B：不，也许让他们自己去发现会带来更多乐趣。

A：但是人们不可能自己来发现所有一切，必须要把握好平衡。

B：好吧，可是所有的学习不都是一种自我发现的过程吗？最好的老师不都是去培养学生自行思考的吗？

A：某种意义上来说，的确如此。哎呀，我们怎么都哲学兮兮的了。

B：我仍然沉浸在这些不着边际的数学工作所带来的成就感中，它现在给我了如此的启发，但过去我可是很讨厌它的。

A：是的，我也很为它兴奋，这可比嗑药管用。我的意思是，大脑靠它天生的力量就可以模拟它自身。

B：而且也让人心生情愫呢！

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[1] 新月沃土 (Fertile Crescent) 为中东两河流域沿岸之大片垦地，是人类的重要发源地之一。

A. (gazing at the stars) One nice thing about pure mathematics—the things we proved today will never be good for anything, so nobody will be able to use them to make bombs or stuff like that.

B. Right. But we can't be in an ivory tower all the time, either. There are lots of problems in the world, and the right kind of math might help to solve them. You know, we've been away from newspapers for so long, we've forgotten all the problems.

A. Yeah, sometimes I feel guilty about that ...

Maybe the right kind of mathematics would help solve some of these problems, but I'm worried that it could also be misused.

B. That's the paradox, and the dilemma. Nothing can be done without tools, but tools can be used for bad things as well as good. If we stop creating things, because they might be harmful in the wrong hands, then we also stop doing useful things.

A. Okay, I grant you that pure mathematics isn't the answer to everything. But are you going to abolish it entirely just because it doesn't solve the world's problems?

B. Oh no, don't misunderstand me. These past few days have shown me that pure mathematics is beautiful—it's an art form like poetry or painting or music, and it turns us on. Our natural curiosity has to be satisfied. It would destroy us if we couldn't have some fun, even in the midst of adversity.

A. Bill, it's good to talk with you like this.

B. I'm enjoying it too. It makes me feel closer to you, and sort of peaceful.

A：(凝望星空)纯数学的一个好处在于——我们今天证明出来的成果没有任何的实际用途，所以没有人会用它来制造炸弹或是类似的东西。

B：没错。但是我们也无法一直呆在象牙塔里。这个世界有着种种的问题，而某种正确的数学就有可能为解决它们提供帮助。你也知道，我们好长时间不读书看报，已经忘却了好多问题了。

A：是啊，我有时会对此感觉一些愧疚……

也许某种正确的数学可以有助于解决这些问题中的一部分，但是我也担心它同样可能被误用。

B：这就是矛盾，这就是两难。没有工具，什么都干不了，但是工具既可以用来干好事，也可以用来干坏事。如果我们仅仅因为可能被居心不良的人掌握而被用来干坏事，就停止创造新事物，那末我们同时也就停止制造有用的东西了。

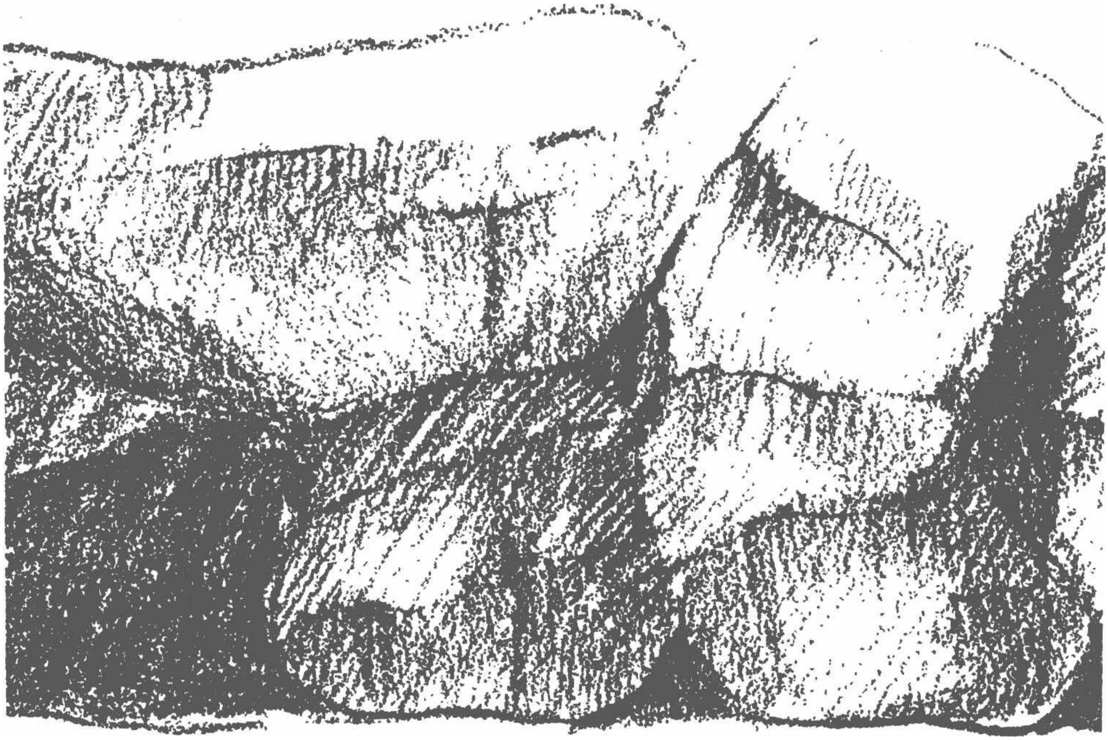
A：OK，我敢向你保证，纯数学不是解决所有问题的办法。但是你不会因为它不能解决这个世界的问题，就把它完全抛弃了吧？

B：哦，不，请不要误解我的意思。过去的几天里，我已经感受到了纯数学的美——它是一种类似于诗歌、绘画和音乐的艺术形式，并且启发了我们。我们天生的好奇心得到了满足。如果一直不能找到用来取乐的事情，这会毁了我们的，即使在苦难中也不例外。

A：Bill，能和你这样交谈我感觉好开心。

B：我也很开心。这种谈话拉近了我们的距离，而且让我感觉到了内心的安宁。

# 12 DISASTER



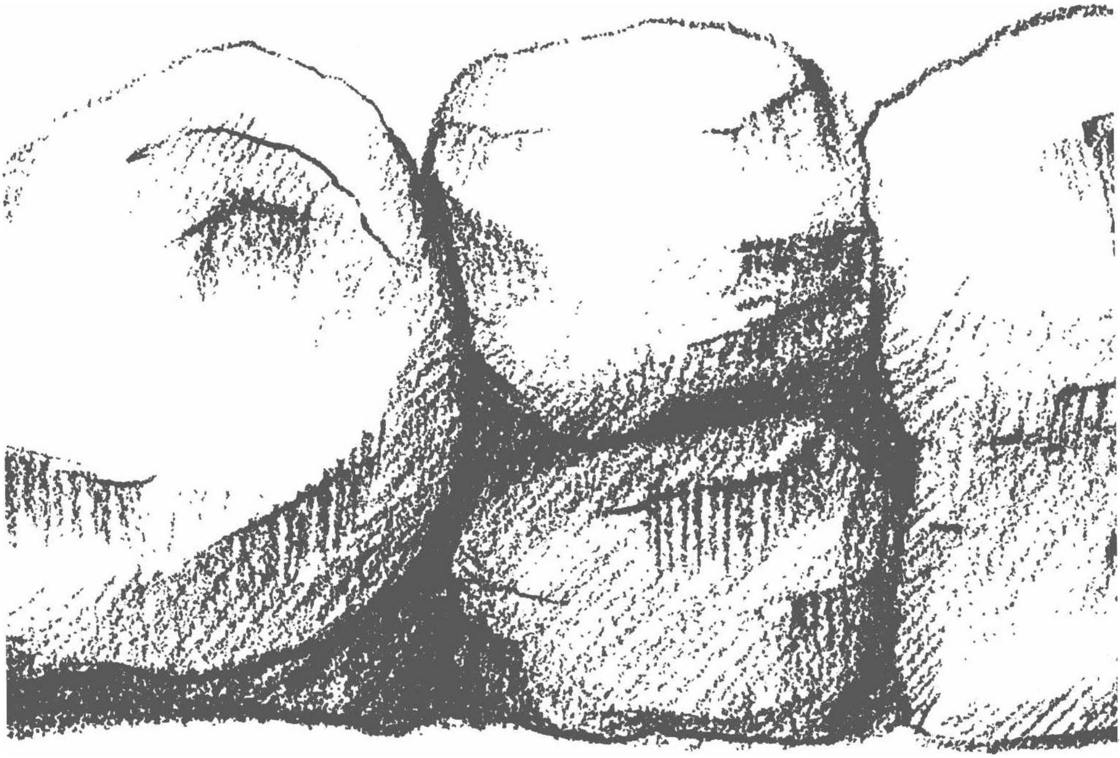
B. Are you awake already?

A. About an hour ago I woke up and realized that there's a big, gaping hole in what we thought we proved yesterday.

B. No!

A. Yes, I'm afraid so. We forgot to prove that $x + y$ is a *number*.

# 12 灾　难



B：这么早你就醒啦？

A：一小时前，我醒来时意识到，在我们以为昨天已经证明好了的东西里有一个巨大的漏洞。

B：不是吧！

A：恐怕是的。我们忘记证明 $x+y$ 是个数了。

B. You're kidding. Of course it's a number, it's the sum of two numbers! Oh wait, I see ... we have to check that rule (1) is satisfied.

A. Yes, the definition of addition isn't legitimate unless we can prove that $X_L + y < X_R + y$, and $X_L + y < Y_R + x$, and $Y_L + x < X_R + y$, and $Y_L + x < Y_R + x$.

B. These would follow from (T13) and (T14), but ... I see your point, we proved (T13) and (T14) assuming that the sum of two numbers is a number. How did you ever think of this problem?

A. Well, that's kind of interesting. I was wondering what would happen if we defined addition like this:

$$x \oplus y = (X_L \oplus Y_L,\ X_R \oplus Y_R).$$

I called this $\oplus$ because it wasn't obviously going to come out the same as $+$. But it was pretty easy to see that $\oplus$ was a commutative and associative operation, so I wanted to see what it turned out to be.

B. I see; the sum of $x$ and $y$ lies between $X_L + Y_L$ and $X_R + Y_R$, so this definition might turn out to be simpler than Conway's.

A. But my hopes were soon dashed, when I discovered that

$$0 \oplus x = 0$$

for all $x$.

B. Ouch! Maybe $\oplus$ means multiplication?

A. Then I proved that $1 \oplus x = 1$ for all $x > 0$, and $2 \oplus x = 2$ for all $x > 1$, and $3 \oplus x = 3$ for all $x > 2$, and ...

B. I see. For all positive integers $m$ and $n$, $m \oplus n$ is the *minimum* of $m$ and $n$. That's commutative and associative, all right. So your $\oplus$ operation *did* turn out to be interesting.

B：你在说笑吧。这当然是个数，它不是两个数的和吗？等一下，我明白了……我们得看一下它是否满足规则 (1)。

A：对，加法规则不一定合法，除非我们可以证明 $X_L + y < X_R + y$、$X_L + y < Y_R + x$、$Y_L + x < X_R + y$，以及 $X_L + y < Y_R + x$。

B：这些可以从 (T13) 和 (T14) 推出吧，但是……我懂你的意思了。我们证明 (T13) 和 (T14) 的过程中已经做了两数之和仍为数的假定。你对这个问题有何高见？

A：嗯，这个问题有点儿意思。我在想，如果我们这样来定义加法的话，又会如何：

$$x \oplus y = (X_L \oplus Y_L, X_R \oplus Y_R)$$

我把这个称为 $\oplus$，因为它和 $+$ 的结果不一定一样。但是显而易见，$\oplus$ 运算是可交换、可结合的，所以我想看看这个运算究竟能算出些什么。

B：明白。$x$ 和 $y$ 的和应该位于 $X_L + Y_L$ 和 $X_R + Y_R$ 之间，所以这个定义也许是比 Conway 的那个要简单些。

A：可是我的希望很快就破灭了，因为我发现对任意的 $x$ 有

$$0 \oplus x = 0$$

B：哟！难道 $\oplus$ 代表乘法么？

A：接下来我证明了对于所有满足 $x > 0$ 的数，有 $1 \oplus x = 1$；以及对于所有满足 $x > 1$ 的数，有 $2 \oplus x = 2$；以及对于所有满足 $x > 2$ 的数，有 $3 \oplus x = 3$；以及……

B：我懂了。对于所有的正数 $m$ 和 $n$ 而言，$m \oplus n$ 是 $m$ 和 $n$ 中较小的那个数。这也是满足交换律和结合律的。所以，你发明的这个 $\oplus$ 好像真的还有点儿意思。

A. Yes, and $\frac{1}{2} \oplus \frac{1}{2} = \frac{1}{2}$. But then I tried $(-\frac{1}{2}) \oplus \frac{1}{2}$, and I was stopped cold.

B. You mean ...?

I see, $(-\frac{1}{2}) \oplus \frac{1}{2} = \big(\{(-1) \oplus 0\}, \{0 \oplus 1\}\big)$, which is $\big(\{0\}, \{0\}\big)$.

A. And that's *not* a number. It breaks rule (1).

B. So your definition of $\oplus$ wasn't legit.

A. And I realized that you can't just go making arbitrary definitions; they have to be proved consistent with the other rules too. Another problem with $\oplus$ was, for example, that $\big(\{-1\}, \emptyset\big) \equiv 0$ but $\big(\{-1\}, \emptyset\big) \oplus 1 \not\equiv 0 \oplus 1$.

B. Okay, $\oplus$ is out, but I suppose we can fix up the *real* definition of $+$.

A. I don't know; what I've just told you is as far as I got. Except that I thought about *pseudo-numbers.*

B. Pseudo-numbers?

A. Suppose we form $(X_L, X_R)$ when $X_L$ is not necessarily $< X_R$. Then rule (2) can still be used to define the $\le$ relation between such pseudo-numbers.

B. I see ... like $\big(\{1\}, \{0\}\big)$ turns out to be less than 2.

A. Right. And I just noticed that our proof of the *transitive* law (T1) didn't use the $\ngeq$ part of rule (1), so that law holds for pseudo-numbers too.

B. Yes, I remember saying that the full rule (1) wasn't used until (T2). That seems like a long time ago.

A. Now get ready for a shock. The pseudo-number $\big(\{1\}, \{0\}\big)$ is neither $\le 0$ nor $\ge 0$!

B. Far out!

A. Yes, I think I can prove that $\big(\{1\}, \{0\}\big)$ is $\le$ a number $y$ if and only if $y > 1$, and it is $\ge$ a number $x$ if and only if $x < 0$. It's not related at all to any numbers between 0 and 1.

A：对，还有 $\frac{1}{2} \oplus \frac{1}{2} = \frac{1}{2}$。但是接下来，我尝试了 $(-\frac{1}{2}) \oplus \frac{1}{2}$，顿时心灰意冷了。

B：你的意思是……？

我明白了，$(-\frac{1}{2}) \oplus \frac{1}{2} = \big(\{(-1) \oplus 0\}, \{0 \oplus 1\}\big)$，也就是 $\big(\{0\}, \{0\}\big)$。

A：但这并非一个数。它不满足规则 (1)。

B：所以你这个 $\oplus$ 的定义并不合法。

A：然后我意识到，并不能任意地发明定义。他们必须和其他规则保持相容才可以。关于 $\oplus$，还有一个问题，比如，$\big(\{-1\}, \emptyset\big) \equiv 0$，可是 $\big(\{-1\}, \emptyset\big) \oplus 1 \neq 0 \oplus 1$。

B：好吧，$\oplus$ 出局了，但是我觉得我们肯定能研究出 $+$ 的*真正*定义。

A：我不确定。我刚才告诉你的就是我刚才研究出来的所有东西了。除此之外，我还冒出了一个叫做*伪数*的想法。

B：伪数？

A：假设我们造出一个数对 $(X_L, X_R)$，但是 $X_L$ 不一定 $< X_R$。这样的话，规则 (2) 仍然可以定义伪数之间的关系。

B：我明白了……比如，$\big(\{1\}, \{0\}\big)$ 就比 2 小。

A：就是这样。而且，我才发现，我们在*传递律* (T1) 的证明过程中，并没有用到规则 (1) 的 $\not\geq$ 部分，所以这条规则同样也适用于伪数。

B：是的，我记得我说过，规则 (1) 直到在推导 (T2) 之前，从来没有用过一次。即使是那件事，也已经过去好久了吧。

A：现在我们得准备震惊一下了。伪数 $\big(\{1\}, \{0\}\big)$ 既不 $\leq 0$ 也不 $\geq 0$！

B：太震撼了！

A：没错，我想我可以证明当且仅当 $\big(\{1\}, \{0\}\big)$ 满足 $\leq$ 数 $y$ 当且仅当 $y > 1$，以及满足 $\geq$ 数 $x$ 当且仅当 $x < 0$。但它与 0 和 1 之间的任何数都没有关系。

B. Where's the pencil? I want to check that out. ... I think you're right. This is fun, we're proving things about quantities that don't even exist.

A. Well, do pseudo-numbers exist any less than Conway's numbers? What you mean is, we're proving things about quantities that are purely conceptual, without familiar real-world counterparts as aids to understanding. ... Remember that $\sqrt{-1}$ was once thought of as an "imaginary" number, and $\sqrt{2}$ wasn't even considered to be "rational."

B. Conway's rule for adding ordinary numbers also gives us a way to add pseudo-numbers. I wonder what this leads to? If $x = (\{1\}, \{0\})$, then $1 + x$ is ... $(\{2\}, \{1\})$.

A. And $x + x$ is $(\{1 + x\}, \{x\})$, a second-order pseudo-number.

B. Pure mathematics is a real mind-expander.

But did you notice that $(\{1\}, \{0\})$ isn't even $\leq$ itself?

A. Let's see; $x \leq x$ means that $X_L < x < X_R$, so this could only be true if $X_L < X_R$.

No, wait, we aren't allowed to use "$<$" in place of "$\not\geq$" for pseudo-numbers, since (T4) isn't true in general. We have to go back to the original rule (2), which says that $x \leq x$ if and only if $X_L \not\geq x$ and $x \not\geq X_R$. So $(\{1\}, \{0\})$ *is* $\geq$ itself after all.

B. *Touché*! I'm glad I was wrong, since every $x$ ought to be like itself, even when it's a pseudo-number.

A. Maybe there is some more complicated pseudo-number that isn't $\leq$ itself. It's hard to visualize, because the sets $X_L$ and $X_R$ might include pseudo-numbers too.

B. Let's look back at our proof of (T3) and see if it breaks down.

A. Good idea. ... Hey, the same proof goes through for all pseudo-numbers: $x$ is *always* like $x$.

B：我的铅笔呢？我想验证一下……我觉得你说得没错。这很有意思，我们在证明有关某个量的事情，但这个量根本不存在。

A：可是，伪数的存在比起 Conway 所定义的数而言，有任何低它一等的地方吗？你真正想说的是，我们在证明纯粹概念的量，而并没有任何现实世界中的对应物来帮助我们理解它们……不过，$\sqrt{-1}$ 一度被认为是所谓"虚"数，而 $\sqrt{2}$ 甚至曾经不被当做是"有理"的呢。

B：Conway 用以引入普通数的法则，同时也给了我们一种引入伪数的办法。我在想，如果一直这样推导下去可以得到什么？如果 $x = (\{1\}, \{0\})$，那么 $1 + x$ 的结果就是……$(\{2\}, \{1\})$。

A：还有，$x + x$ 的结果是 $(\{1 + x\}, \{x\})$，这是个二阶伪数。

B：纯粹数学真的是一种观念扩展器。

但你没发现吗？$(\{1\}, \{0\})$ 甚至不 $\leq$ 它自己！

A：让我瞧瞧。$x \leq x$ 的意思是 $X_L < x < X_R$，所以当且仅当 $X_L < X_R$ 才成立。

不过，等等，对于伪数来说，在"$\not\geq$"的地方使用"$<$"来代替是不能被允许的，因为 (T4) 的成立不具一般性。我们必须回到原始规则 (2)，它规定 $x \leq x$ 当且仅当 $X_L \not\geq x$ 且 $x \geq X_R$。所以，$(\{1\}, \{0\})$ 到底还是 $\geq$ 它自己的啊。

B：高见！幸亏我是错的，因为每个 $x$ 都应该相似于它自己，即使是伪数也不应该例外。

A：也许存在一些复杂的伪数，它们不 $\leq$ 自己。说起来很不直观，但 $X_L$ 和 $X_R$ 也可能包含伪数。

B：那就来看看我们对于 (T3) 的证明是不是不成立了。

A：好主意……嘿，我们的证明对于所有的伪数都畅行无虞：$x$ 恒相似于 $x$。

B. This is great but I'm afraid it's taking us away from the main problem, whether or not + is well-defined.

A. Well, our proofs of $x + y = y + x$ and $x + 0 = x$ work for pseudo-numbers as well as numbers, and so does our proof of the associative law. If the inequality theorems (T13) and (T14) also go through for pseudo-numbers, then + will be well-defined.

B. I see, that's beautiful! So far we've established (T1), (T3), (T5), (T6), (T9), (T10), and (T11) for all pseudo-numbers. Let's look at (T13) again.

A. But I'm afraid ... uh, oh, Bill! We were too gullible yesterday in our acceptance of that day-sum proof for (T13) and (T14); it was too good to be true.

B. What do you mean?

A. We were proving that $Z_L + x < y + z$ by induction, right? Well, to get this it takes two steps, first $Z_L + x \leq Z_L + y$ and then $Z_L + y < z + y$. Induction gives us the first part all right, but the second part involves $(z_L, z, y)$, which might have a *larger* day-sum than $(x, y, z)$.

B. So we really blew it. Conway would be ashamed of us.

A. Good thing we didn't see this yesterday, or it would have spoiled our day.

B. I guess it's back to the drawing boards ... but hey, we've *gotta* eat some breakfast.

B：这确实不错，但是恐怕它让我偏离讨论主题愈发地远了：我们是在讨论 + 是否有一个合式的定义。

A：好吧，我们对于 $x+y=y+x$ 和 $x+0=x$ 的证明对于伪数来说，就像对于数一样地成立，结合律也一样。如果不等式定理 (T13) 和 (T14) 对于伪数也同样成立的话，+ 的定义就是合式的了。

B：我明白了，太漂亮了！现在我们已经把 (T1)、(T3)、(T5)、(T6)、(T9)、(T10)、(T11) 对于伪数建立起来了。让我们再重新审视一下 (T13)。

A：但我担心……哟！Bill，我们昨天对于 (T13) 和 (T14) 的创造日之和证明太过轻信地就接受了，这个结果好得有点儿过头了。

B：你是什么意思呢？

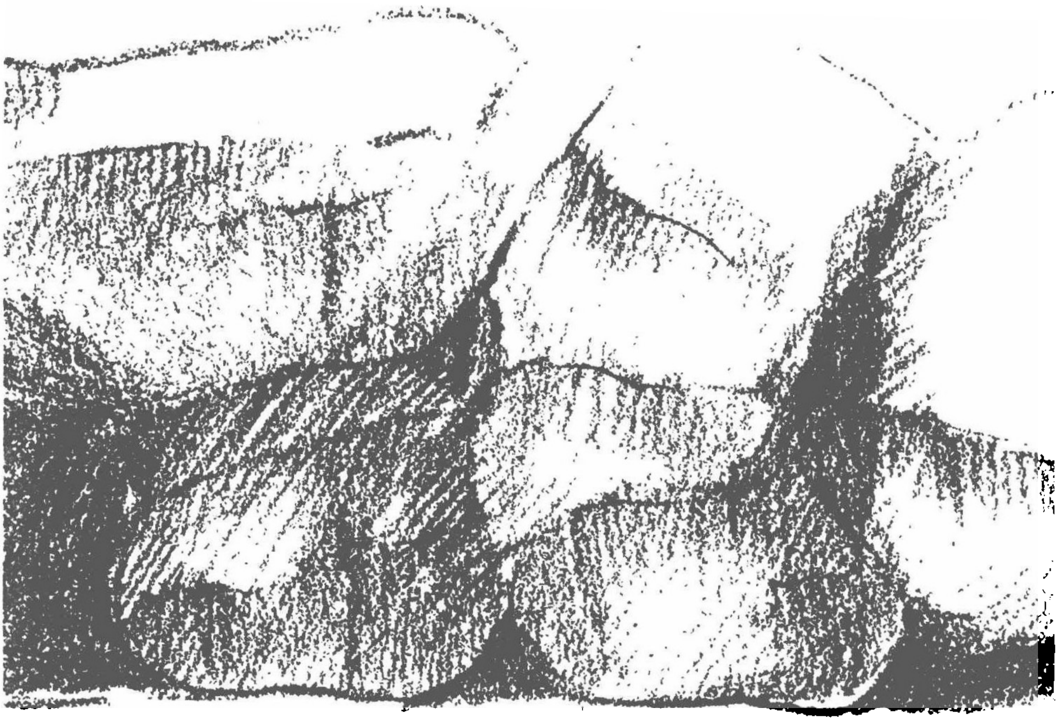
A：我们是用归纳法来证明 $Z_L+x=y+z$ 的，没错吧？好，要得到这个结果，要分两部分进行，先证明 $Z_L+x\le Z_L+y$，再证明 $Z_L+y<z+y$。归纳法用以证明第一部分是没啥问题，但是第二部分涉及了 $(z_L,z,y)$，它可能比 $(x,y,z)$ 的创造日之和更大。

B：所以我们真的是搞砸了。Conway 会为我们感觉害臊的吧。

A：幸亏我们昨天没发现这个问题，要不然肯定一整天就毁了。

B：看来我们只能另起炉灶才成……可是，嘿，我们真的得弄点儿早饭吃了。

# 13 RECOVERY



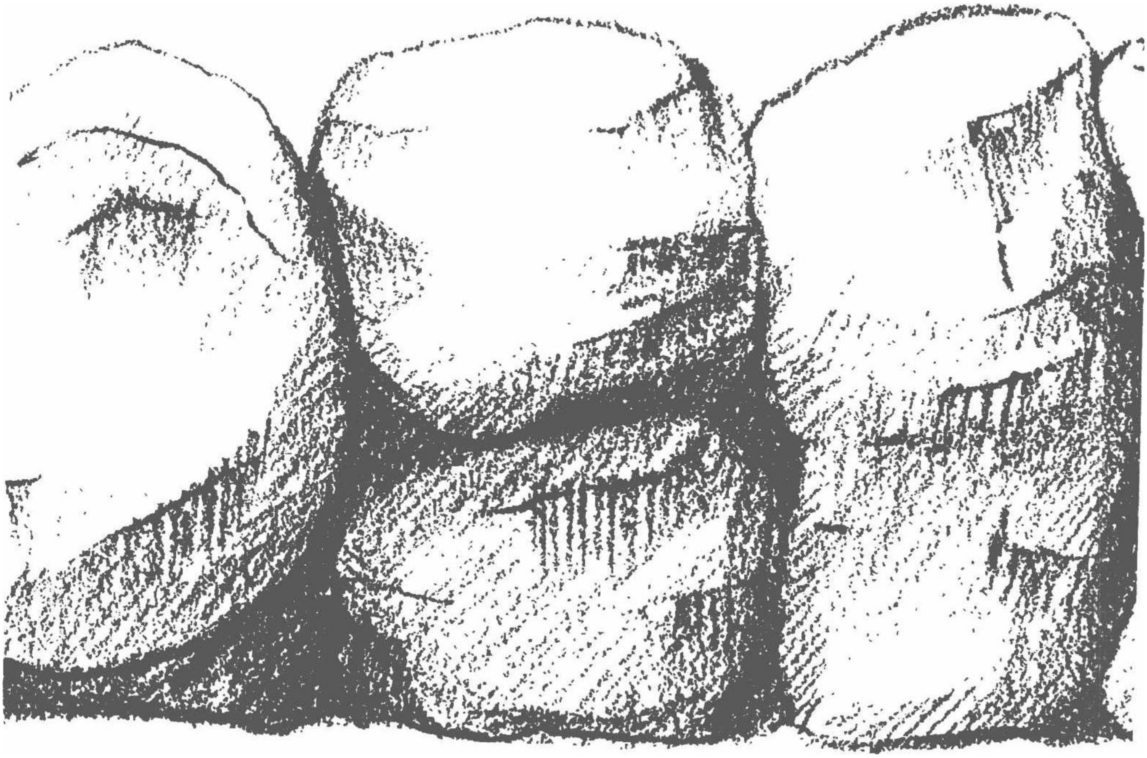
A. We've missed lunch, Bill.

B. (pacing the ground) Have we? This stupid problem is driving me up the wall.

A. Just staring at this paper isn't helping us any, either. We need a break; maybe if we ate something—

B. What we really need is a new idea. Gimme an idea, Alice.

# 13 平 复



A：我们已经错过午饭时间啦，Bill。

B：（踱来踱去）真的吗？这个莫名其妙的问题简直把我逼疯了。

A：盯着纸片看不也于事无补吗？我们需要休息一下，也许吃点东西的话……

B：我们真正需要的是个好思路。给我点儿思路啊，Alice。

A. (beginning to eat) Well, when we were going around in circles like this before, how did we break out? The main thing was to use induction, I mean to show that the proof in one case depended on the truth in a *previous* case, which depended on a still previous case, and so on, where the chain must eventually come to an end.

B. Like our day-sum argument.

A. Right. The other way we broke the circle was by proving *more* than we first thought we needed. I mean, in order to keep the induction going, we had to keep proving several things simultaneously.

B. Like when you combined (T13) and (T14). Okay, Alice, right after lunch I'm going to sit down and write out the total picture, everything we need to prove, and perhaps even more. And I'm going to try and prove everything simultaneously by induction. The old battering-ram approach. If that doesn't work, nothing will.

A. That sounds hard but it's probably the best way. Here, have some oat cakes.

. . . . . . . .

B. Okay, here we go. We want to prove three things about numbers, and they all seem to depend on each other.

| | |
|---|---|
| $\mathrm{I}(x, y)$: | $x + y$ is a number. |
| $\mathrm{II}(x, y, z)$: | if $x \leq y$, then $x + z \leq y + z$. |
| $\mathrm{III}(x, y, z)$: | if $x + z \leq y + z$, then $x \leq y$. |

A：（开始吃东西）那么，以前我们如果陷入这样的怪圈时，是如何找到出路的？主要的办法是运用归纳法，我的意思是，说明在某种情况下给出的证明依赖于前一种情况，而后者又依赖于更前一种情况，依此类推，这条证明链肯定有个终点。

B：就像我们的创造日之和方法。

A：对。另一种找到出路的办法是证明出比我们最初认为所必需的还要更多的结论。我是说，为了能让归纳进行下去，我们就得一直同时证明若干个结论。

B：就像你把 (T13) 和 (T14) 组合起来时所做的那样。好吧，Alice，吃完午饭我得马上坐下来把整幅图景写下来，写出我们需要证明的全部内容，可能还有更多。并且，我打算试着运用归纳法来同时证明所有结论。也就是那种能让我们破墙而出的工具。如果这个都行不通，那也就无计可施了。

A：听起来很困难呢，但是可能这也是最好的办法了。来点儿燕麦糕。

…………

B：好，现在开工。我们意欲证明的是有关数的三件事，并且它们看起来是相互依赖的：

$$\begin{aligned} \mathrm{I}(x,y): &\quad x+y \text{ 是数。} \\ \mathrm{II}(x,y,z): &\quad \text{若 } x \le y\text{，则 } x+z \le y+z\text{。} \\ \mathrm{III}(x,y,z): &\quad \text{若 } x+z \le y+z\text{，则 } x \le y\text{。} \end{aligned}$$

Now if I'm not mistaken, the proof of $\mathrm{I}(x, y)$ will follow if we have previously proved

$$\mathrm{I}(X_L, y), \quad \mathrm{I}(x, Y_L), \quad \mathrm{I}(X_R, y), \quad \mathrm{I}(x, Y_R),$$
$$\mathrm{III}(X_R, X_L, y),$$
$$\mathrm{III}(x, X_L, y), \quad \mathrm{II}(y, Y_R, x),$$
$$\mathrm{III}(y, Y_L, x), \quad \mathrm{II}(x, X_R, y),$$
$$\mathrm{III}(Y_R, Y_L, x).$$

For example, we have to prove among other things that $X_L + y < Y_R + x$. In other words, for all $x_L$ in $X_L$ and $y_R$ in $Y_R$ we should have previously established that $x_L + y < y_R + x$. Now $\mathrm{III}(x, x_L, y)$ and (T3) show that $x_L + y < x + y$, and $\mathrm{II}(y, y_R, x)$ shows that $y + x \leq y_R + x$. Right?

A. It looks good; except I don't see why you included those first four, $\mathrm{I}(X_L, y)$ through $\mathrm{I}(x, Y_R)$. I mean, even if $x_L + y$ wasn't a number, that wouldn't matter; all we really need to know is that $x_L$ and $y$ themselves are numbers. After all, $<$ and $\leq$ are defined for pseudo-numbers, and the transitive laws work too.

B. No, rule (1) says that elements of the left part like $x_L + y$ have to be numbers. Anyway it doesn't really matter, because if we're proving $\mathrm{I}(x, y)$ we can assume $\mathrm{I}(x_L, y)$ and so on for free; induction takes care of them.

A. It's complicated, but keep going, this looks promising.

B. This approach *has* to work, or we're sunk. Okay, the proof of $\mathrm{II}(x, y, z)$, namely (T13), will follow if we have previously proved

$$\mathrm{III}(y, X_L, z),$$
$$\mathrm{II}(x, y, Z_L), \quad \mathrm{III}(z, Z_L, y),$$
$$\mathrm{III}(Y_R, x, z),$$
$$\mathrm{II}(x, y, Z_R), \quad \mathrm{III}(Z_R, z, x).$$

如果我没搞错的话，若欲使 $\mathrm{I}(x, y)$ 成立，必得先证明以下结论

$$\mathrm{I}(X_L, y), \quad \mathrm{I}(x, Y_L), \quad \mathrm{I}(X_R, y), \quad \mathrm{I}(x, Y_R),$$
$$\mathrm{III}(X_R, X_L, y),$$
$$\mathrm{III}(x, X_L, y), \quad \mathrm{II}(y, Y_R, x),$$
$$\mathrm{III}(y, Y_L, x), \quad \mathrm{II}(x, Y_R, y),$$
$$\mathrm{III}(Y_R, Y_L, x).$$

举例来说，除了这些我们还必须证明 $X_L + y < Y_R + x$。换言之，对于 $X_L$ 中的所有 $x_L$ 以及 $Y_R$ 中的 $y_R$，我们本该先证出 $x_L + y < y_R + x$ 才对。因为 $\mathrm{III}(x, x_L, y)$ 和 (T3) 说明了 $x_L + y < x + y$，而 $\mathrm{II}(x, y_R, y)$ 说明了 $y + x < y_R + x$。对吗？

A：看上去没什么问题，只有一点我看不太明白，就是为什么你要把最开始的四个条件，就是从 $\mathrm{I}(X_L, y)$ 到 $\mathrm{I}(x, Y_R)$，给包含进来呢？我是想说，即使 $x_L + y$ 并不是数，也不要紧的。我们只需确定 $x_L$ 和 $y$ 自身都是数即可。说到底，$<$ 和 $\leq$ 已经对于伪数有了定义，而且传递律对它们也成立了。

B：不对，规则 (1) 明确规定了，像 $x_L + y$ 这样的左集元素必须是数。虽然这并不真正会造成什么影响，因为如果我们要证明 $\mathrm{I}(x, y)$，我们可以自由地假定 $\mathrm{I}(x_L, y)$，等等。自有归纳法来保证其成立。

A：有点儿玄乎，但你还是继续说下去，看上去还挺有希望。

B：这条路必须要走得通，否则我们就全完了。好，现在看 $\mathrm{II}(x, y, z)$，也就是 (T13) 的证明，它成立的前提是先满足以下条件

$$\mathrm{III}(y, X_L, z),$$
$$\mathrm{II}(x, y, Z_L), \quad \mathrm{III}(z, Z_L, y),$$
$$\mathrm{III}(Y_R, x, z),$$
$$\mathrm{II}(x, y, Z_R), \quad \mathrm{III}(Z_R, z, x).$$

That's curious — this one really *doesn't* require I$(x, y)$. How come we thought we'd have to prove that the sum of two numbers is a number, before proving (T13)?

A. That was before we knew much about pseudo-numbers. It's strange how a fixed idea will remain as a mental block!

Remember? This was the first reason we said it was going to be hard to prove $x + y$ is a number, because we thought (T13) depended on this. After learning that pseudo-numbers satisfy the transitive laws, we forgot to reconsider the original source of trouble.

B. So at least this big-picture method is getting us somewhere, if only because it helps organize our thoughts.

Now it's two down and one to go. The proof of III$(x, y, z)$ depends on knowing

$$\text{II}(y, X_L, z),$$
$$\text{II}(Y_R, x, z).$$

A. Again, I$(x, y)$ wasn't required. So we can simply prove (T13) and (T14) without worrying whether or not $x + y$ is a number.

B. I see — then later, $x + y$ will turn out to be a number, because of (T13) and (T14). Great!

A. Now II and III depend on each other, so we can combine them into a single statement like we did before.

B. Good point. Let's see, if I write IV$(x, y, z)$ to stand for the combined statement "II$(x, y, z)$ and III$(x, y, z)$," my lists show that IV$(x, y, z)$ depends on

$$\text{IV}(y, X_L, z), \qquad \text{IV}(x, y, Z_L), \qquad \text{IV}(z, Z_L, y),$$
$$\text{IV}(Y_R, x, z), \qquad \text{IV}(x, y, Z_R), \qquad \text{IV}(Z_R, z, x).$$

很不寻常——这一条的确不要求 $\mathrm{I}(x,y)$。我们怎么会想到，在证明 (T13) 之前，要去证明两个数的和亦为数的？

A：那是因为在那个时候我们对伪数还知之甚少。思维定势真的害死人哪，它总成为思想的阻力！

还记得吗？一开始，我们说证明 $x+y$ 是数很困难，因为我们觉得 (T13) 依赖于此。而知道了伪数也满足传递律以后，我们好了疮疤，却忘了疼。

B：至少全景框架法已经给我们带来了一些进展，即使它只是帮我们整理了一下思路。

现在已经发球两次，尚余一击。$\mathrm{III}(x,y,z)$ 的证明依赖于以下的结论

$$\mathrm{II}(y, X_L, z),$$
$$\mathrm{II}(Y_R, x, z).$$

A：同样地，并没有要求 $\mathrm{I}(x,y)$。所以，我们可以直接证明 (T13) 和 (T14)，而并不用操心 $x+y$ 到底是不是数，好棒。

B：我懂了——稍后，$x+y$ 就肯定是数了，因为由 (T13) 和 (T14) 可以推得这个结论。真是太好了！

A：这么一来，II 和 III 就互相依赖了，所以我们可以把它们结合成一个单独的命题，就像我们以前所做的那样。

B：说得好。我们来看看，如果我把新命题写作 $\mathrm{IV}(x,y,z)$，表示组合命题“$\mathrm{II}(x,y,z)$ 且 $\mathrm{III}(x,y,z)$”，我上面的列表就说明，$\mathrm{IV}(x,y,z)$ 依赖于

$$\mathrm{IV}(y,X_L,z),\qquad \mathrm{IV}(x,y,Z_L),\qquad \mathrm{IV}(x,Z_L,y),$$
$$\mathrm{IV}(Y_R,x,z),\qquad \mathrm{IV}(x,y,Z_R),\qquad \mathrm{IV}(Z_R,z,x).$$

I think it was a good idea to introduce this new notation, like $I(x, y)$ and so on, because it makes the patterns become clearer. Now all we have to do is find some way to rig up an induction hypothesis that goes from these six things to $IV(x, y, z)$.

A. But uh-oh, it doesn't work. Look, $IV(x, y, z)$ depends on $IV(z, z_L, y)$, which depends on $IV(y_R, y, z)$, which depends on $IV(z, z_L, y)$ again; we're in a loop. It's the same stupid problem I noticed before, and now we know it's critical.

B. (pounding the dirt) *Oh no!* ... Well, there's one more thing I'll try before giving up. Let's go all the way and prove a more general version of (T13):

$$V(x, x', y, y'): \qquad \text{if } x \le x' \text{ and } y \le y', \text{ then } x + y \le x' + y'.$$

This is what we really are using in our proofs, instead of doing two steps with (T13). And it's symmetrical; that might help.

A. We'll also need a converse, generalizing (T14).

B. I think what we need is

$$VI(x, x', y, y'): \qquad \text{if } x + y \ge x' + y' \text{ and } y \le y', \text{ then } x \ge x'.$$

A. Your notation, primes and all, looks very professional.

B. (concentrating) Thank you. Now the proof of $V(x, x', y, y')$ depends on

$$VI(X_L, x', y, y'),$$
$$VI(Y_L, y', x, x'),$$
$$VI(x, X'_R, y, y'),$$
$$VI(y, Y'_R, x, x').$$

Hey, this is actually easier than the other one, the symmetry is helping.

我觉得引入这种新记法还是挺不错的，就像 $\mathrm{I}(x,y)$ 这样，因为它使得模式清晰地展现出来了。现在我们要做的所有事情就是找出某些方法来凑出个归纳假设，来从这六者推得 $\mathrm{IV}(x,y,z)$。

A：可是，哎哟，行不通的。你看，$\mathrm{IV}(x,y,z)$ 依赖于 $\mathrm{IV}(z,z_L,y)$，$\mathrm{IV}(z,z_L,y)$ 又依赖于 $\mathrm{IV}(y_R,y,z)$，而 $\mathrm{IV}(y_R,y,z)$ 又反过来依赖于 $\mathrm{IV}(x,y,z)$：我们又陷于循环论证了。这正是我先前遇到的愚蠢问题，现在我们了解到它十分要紧了。

B：（拍拍身上的灰尘）不会吧！……在放弃之前，我要再尝试一件事。让我们略过所有内容，来证明 (T13) 的一个更一般化的版本：

$$\mathrm{V}(x,x',y,y'):\qquad \text{若 } x\le x' \text{ 且 } y\le y'\text{，则 } x+y\le x'+y'\text{。}$$

这种表述是我们真正用在证明中的，而不是在 (T13) 中采取两步走的办法。并且，它是对称的。这可能会有所帮助。

A：我们同样也需要它的逆命题，一个一般化的 (T14)。

B：我想，我们所需要的是：

$$\mathrm{VI}(x,x',y,y'):\qquad \text{若 } x+y\ge x'+y' \text{ 且 } y\le y'\text{，则 } x\ge x'\text{。}$$

A：你的记法、符号和所有的一切，看起来都好专业。

B：（集中精力思考）多谢夸奖。现在 $\mathrm{V}(x,x',y,y')$ 的证明依赖于

$$\mathrm{VI}(X_L,x',y,y'),$$
$$\mathrm{VI}(Y_L,y',x,x'),$$
$$\mathrm{VI}(x,X'_R,y,y'),$$
$$\mathrm{VI}(y,Y'_R,x,x').$$

嘿，这可比前面那个容易多了，对称性帮上忙了。

Finally, to prove $\mathrm{VI}(x, x', y, y')$, we need ... the suspense is killing me, I can't think ...

$$\mathrm{V}(x, X'_L, y, y'), \qquad \mathrm{V}(X_R, x', y, y').$$

A. (jumping up) Look, a ***day-sum*** argument, applied to the combination of V and VI, now finishes the induction!

B. (hugging her) We've won!

A. Bill, I can hardly believe it, but our proof of these two statements actually goes through for all *pseudo-numbers* $x$, $x'$, $y$, and $y'$.

B. Alice, this has been a lot of work, but it's the most beautiful thing I ever saw.

A. Yes, we spent plenty of energy on what we both took for granted yesterday.

I wonder if Conway himself had a simpler way to prove those laws? Maybe he did, but even so I like ours because it taught us a lot about proof techniques.

B. Today was going to be the day we studied multiplication.

A. We'd better not start it now, it might ruin our sleep again. Let's just spend the rest of the afternoon working out a proof that $-x$ is a number, whenever $x$ is.

B. Good idea, that should be easy now. And I wonder if we can prove something about the way negation acts on pseudo-numbers?

最后，要证明 VI$(x, x', y, y')$，我们需要……迫不及待的心情，已经令我无法思考了……

$$V(x, X'_L, y, y'), \qquad V(X_R, x', y, y').$$

A：（跳了起来）看哪，创造日之和的讨论，现在被应用在 V 和 VI 的组合上了，这么一来归纳证明就完成了！

B：（拥抱 Alice）我们胜利了！

A：Bill，简直难以置信哦，我们对这两个命题的证明其实对所有的伪数 $x$、$x'$、$y$ 和 $y'$ 都是成立的。

B：Alice，虽然历尽千辛万苦，但是这的确是我所见过的最美的东西。

A：是啊，我们花了这么大的精力在我们昨天觉得是理所当然的事情上。

我在想，Conway 自己是不是有更简单的办法来证明这些定律呢？也许他有，但我却更喜欢我们自己的办法，因为通过这些，我们学到了好多证明技术。

B：今天我们动手来研究乘法吧。

A：最好不要现在吧，不然又要睡不好觉了。我们今天下午的其余时间，就用来证明一下无论 $x$ 是什么数，$-x$ 都是数。

B：好想法，现在这个会是容易的事了吧。我想，我们是不是可以从作用于伪数的反证法中证明出什么来呢？

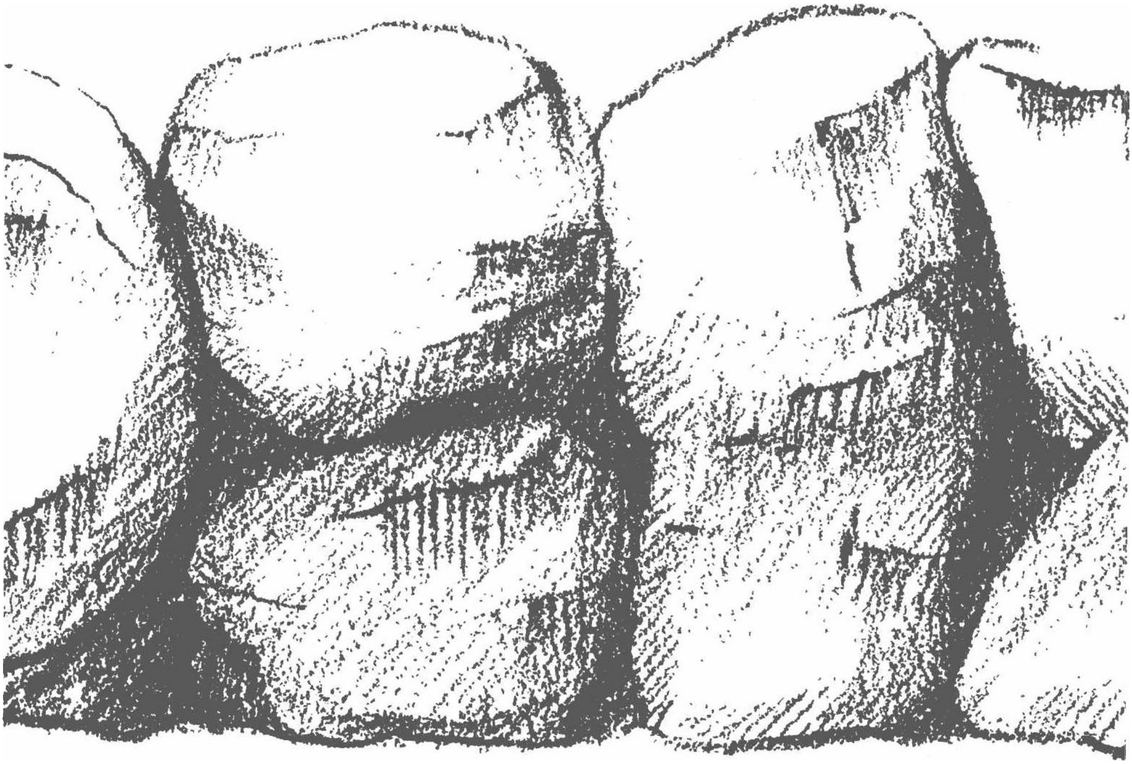
# 14 THE UNIVERSE

B. (stretching) Good morning, love; did you think of any more mistakes in our math, during the night?

A. No, how about you?

B. You *know* I never look for mistakes. But a thought did strike me: Here we're supposed to have rules for creating all the numbers, but actually $\frac{1}{3}$ never appears. Remember, I was expecting to see it on the fourth day, but that number turned out to be $\frac{1}{4}$.

# 14 宇 宙



B：（伸懒腰）早上好，亲爱的。你在咱们的数学中发现其他错误了没，我是说这一夜里？

A：没有啊，你那边想到什么了吗？

B：你懂的，我从来不去挑错儿。可是还真的有一种想法突然来袭：我们应该是已经找到了创建所有数的规则，可实际上 $\frac{1}{3}$ 从来没有出现过。还记得吗？我本来以为它会在第四日出现，但是那个数其实是 $\frac{1}{4}$。

I kind of thought, well, $\frac{1}{3}$ is a little slow in arriving, but it will get here sooner or later. Just now it hit me that we've analyzed all the numbers, but $\frac{1}{3}$ has never showed.

A. All the numbers that are created have a finite representation in the binary number system. I mean like $3\frac{5}{8}$ is 11.101 in binary. And on the other hand, every number with a finite binary representation *does* get created, sooner or later. Like, $3\frac{5}{8}$ was created on the ... eighth day.

B. Binary numbers are used on computers. Maybe Conway was creating a computerized world.

What *is* the binary representation of $\frac{1}{3}$ anyway?

A. I don't know, but it must have one.

B. Oh, I remember, you sort of do long division but with base 2 instead of 10. Let's see ... I get

$$\frac{1}{3} = .0101010101\ldots$$

and so on ad infinitum. It doesn't terminate, that's why it wasn't created.

A. "Ad infinitum." That reminds me of the last part of the inscription. What do you suppose the rock means about ℵ day and all that?

B. It sounds like some metaphysical or religious praise of the number system to me. Typical of ancient writings.

On the other hand, it's sort of strange that Conway was still around and talking, after infinitely many days. "Till the end of time," but time hadn't ended.

A. You're in great voice today.

B. After infinitely many days, I guess Conway looked out over all those binary numbers he had created, and ... Omigosh! I bet he *didn't* stop.

我就在想，$\frac{1}{3}$ 可能只是姗姗来迟，但总会出现的。但是刚才我突然发现，我们已经分析了所有的数，可是 $\frac{1}{3}$ 却始终没有出现。

A：所有被创造的数都在二进制数系中有着有限的表示，我的意思是，$3\frac{5}{8}$ 用二进制表示就是 11.101。另一方面，凡在能用二进制有限地表示出来的数也的确都或早或迟地被创造出来了。就好比 $3\frac{5}{8}$ 是在第……八日被创造的。

B：二进制数是用在计算机中的吧，莫非 Conway 是在创造一个电算化的世界吗？

但是无论如何，$\frac{1}{3}$ 的二进制表示究竟是什么呢？

A：我不知道，但是肯定有的。

B：哦，我想起来了，好像要做长除法，但不是以十为基底，而以二为基底。我们看看，这么一来就得到

$$\frac{1}{3} = .0101010101\ldots$$

依此类推，直至无穷 (ad infinitum)。这个过程永无止境，所以我们才没有见到它被创造出来。

A：“直至无穷”，这个词让我联想起了雕文的最后一部分。你觉得那块岩石上写的“א 日”以及所有那些东西，到底是什么意思呀？

B：我觉得，这像是某种玄学或是宗教对于数系的赞颂。这在古文字中很普遍啊。

另一方面，我又有点儿想不通，为什么 Conway 在过了无穷多日以后，仍然还呆在那儿说个不停。“无穷日逝”，可是时间并没有达到尽头呀。

A：你今天说话很有腔调哟。

B：过了无穷多日以后，我猜想 Conway 检视了所有这些他创造出来的二进制数，然后……天哪，我觉得他并未停歇。

A. You're right! I never thought of it before, but the Stone does seem to say that he went right on. And ... sure, he gets more numbers, too, because for the first time he can choose $X_L$ and $X_R$ to be *infinite* sets.

B. Perhaps time doesn't flow at a constant rate. I mean, to us the days seem like they're of equal length; but from Conway's point of view, as he peers into our universe, they might be going faster and faster in some absolute extra-celestial time scale. Like, the first earth day lasts one heavenly day, but the second earth day lasts only half a heavenly day, and the next is one fourth, and so on. Then, after a total of two heavenly days, zap! Infinitely many earth days have gone by, and we're ready to go on.

A. I never thought of that, but it makes sense. In a way, we're now exactly in Conway's position after infinitely many earth-days went by. Because we really *know* everything that transpired, up until ℵ day.

B. (gesticulating) *Another* plus for mathematics: Our finite minds can comprehend the infinite.

A. At least the countably infinite.

B. But the real numbers are uncountable, and we can even comprehend them.

A. I suppose so, since every real number is just an infinite decimal expansion.

B. Or binary expansion.

A. Hey! I know now what happened on ℵ day — the real numbers were all created!

B. (eyes popping) Migosh. I believe you're right.

A. Sure; we get $\frac{1}{3}$ by taking $X_L$ to be, say,

$$\{.01, .0101, .010101, .01010101, \ldots\}$$

A：你说得对！我以前从来没有想过这个问题，但是岩石上似乎的确说了，他仍然不停进取，并且……没错，他也得到了更多的数，因为他可以破天荒地将 $X_L$ 和 $X_R$ 取为无穷集合。

B：也许时间并不是以恒速流逝的。我的意思是，对于你我来说，每一日好像是等长的。但是站在 Conway 的视角来看，他在俯视我们的宇宙时，以某种绝对来自外星的时间尺度来看，每一日的时间肯定是流逝得越来越快的。就好比，首日的时间是“地上一日、天上一日”，而第二日就成了“地上一日，天上半日”，下一日天上只流逝了四分之一日，依此类推。于是，天上一共只用了两日，搞定！然而此地地上已经过了无穷日，而我们则准备好了继续前行。

A：虽然我从来没有这样想过，但你说得有一定道理。从某种意义上说，我们现在正好就身处 Conway 的位置，时间在地球上的无穷时间流逝之后。因为我们完全清楚了所有发生过的事情，直到第 $\aleph$ 日。

B：（打了个手势）有关数学再多说一句：我们有穷的大脑竟可以理解无穷。

A：至少是可数无穷。

B：实数是不可数的，但我们竟然连它都理解得了。

A：我觉得是，因为任何实数都只是一种数的无穷十进制延伸罢了。

B：或是二进制延伸。

A：嘿！我这下明白在第 $\aleph$ 日发生什么事了——全部实数在这一日被创造出来了！

B：（眼睛睁得快要掉下来）我的天哪。我想你是对的！

A：肯定没错了，我们用下面的集合作为 $X_L$，以获得 $\frac{1}{3}$，就是，

$$\{.01, .0101, .010101, .01010101, \ldots\}$$

in binary notation, and $X_R$ would be numbers that get closer and closer to $\frac{1}{3}$ from above, like

$$\{.1, .011, .01011, .0101011, \ldots\}.$$

B. And a number like $\pi$ gets created in roughly the same way. I don't know the binary representation of $\pi$, but let's say it's

$$\pi = 11.00100100001111\ldots;$$

we get $\Pi_L$ by stopping at every "1,"

$$\Pi_L = \{11.001, 11.001001, 11.00100100001, \ldots\}$$

and $\Pi_R$ by stopping at every "0" and increasing it,

$$\Pi_R = \{11.1, 11.01, 11.0011, 11.00101, \ldots\}.$$

A. There are lots of other sets that could be used for $\Pi_L$ and $\Pi_R$, infinitely many in fact. But they all produce numbers equivalent to this one, because it is the first number created that is greater than $\Pi_L$ and less than $\Pi_R$.

B. (hugging her again) So *that's* what the Conway Stone means when it says the universe was created on $\aleph$ day: The real numbers are the universe.

Have you ever heard of the "big bang" theory the cosmologists talk about? This is it, $\aleph$ day: Bang!

A. (not listening) Bill, there's *another* number also created on $\aleph$ day, a number that's not in the real number system. Take $X_R$ to be empty, and

$$X_L = \{1, 2, 3, 4, 5, \ldots\}.$$

This number is larger than *all* the others.

这是它的二进制记法，而 $X_R$ 中就是那些无限从大数方向逐步逼近 $\frac{1}{3}$ 的数，比如说

$$\{.1, .011, .01011, .0101011, \ldots\}.$$

B：像 $\pi$ 这样的数也可以通过差不多的办法创造出来。我不是十分明白 $\pi$ 用二进制该怎么表示，但是假设它是：

$$\pi = 11.00100100001111\ldots;$$

我们可以通过在每个"1"处停下来的方法得到 $\Pi_L$

$$\Pi_L = \{11.001, 11.001001, 11.00100100001, \ldots\}$$

通过在每个"0"处停下并增 1 的方法得到 $\Pi_R$

$$\Pi_R = \{11.1, 11.01, 11.0011, 11.00101, \ldots\}.$$

A：还有很多其他集合可以用作 $\Pi_L$ 和 $\Pi_R$，事实上有无穷多的可能集合。但是它们产生的数都与这个集合在产生的数方面等价，因为这是创造出来的首个大于 $\Pi_L$ 并小于 $X_R$ 的数。

B：（再次拥抱她）所以，那个就是 Conway 之岩所提及的在第 $\aleph$ 日"宇宙现形"的意思。实数即宇宙。

你有没有听过说宇宙学家们谈论过的"大爆炸"理论？这就是了，第 $\aleph$ 日：砰！

A：（没在听）Bill，在第 $\aleph$ 日还创造出来了另一个数，这是个不在实数系中的数。取 $X_R$ 为空集，且取 $X_L$ 为

$$X_L = \{1, 2, 3, 4, 5, \ldots\}.$$

该数比所有其他数都要大。

B. Infinity! Outa sight!

A. I think I'll denote it by the Greek letter $\omega$ since I always liked that letter. Also $-\omega$ was created, I mean minus infinity.

B. $\aleph$ day was a busy, busy day.

A. Now the *next* day—

B. You mean $\aleph$ wasn't the end!

A. Oh no, why should Conway stop then? I have a hunch he was only barely getting started. The process never stops, because you can always take $X_R$ empty and $X_L$ to be the set of all numbers created so far.

B. But there isn't much else to *do* on the day after $\aleph$, since the real numbers fit together so densely. The noninfinite part of the universe is done now, since there's no room to put any more numbers in between two "adjacent" real numbers.

A. No, Bill; that's what *I* thought too, until you said it just now. I guess it just proves that I like to argue with you. How about taking $X_L = \{0\}$ and $X_R = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$. It's a number *greater* than zero and *less* than all positive real numbers! We might call it $\epsilon$.

B. (fainting) Ulp ... That's okay, I'm all right. But this is almost *too* much; I mean, there's gotta be a limit.

What surprises me most is that your number $\epsilon$ was actually created on $\aleph$ day, *not* the day after, because you could have taken $X_R = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$. Also, there are lots of other crazy numbers in there, like

$$(\{1\}, \{1\tfrac{1}{2}, 1\tfrac{1}{4}, 1\tfrac{1}{8}, 1\tfrac{1}{16}, \ldots\})$$

which is just a hair bigger than 1.

And I suppose there's a number like this right next to all numbers, like $\pi$ ... no, that can't be ...

B：无穷大啊！不可思议！

A：我想，我会采用希腊字母 $\omega$ 作为这个数的记法，因为我一直挺喜欢这个字母。还有 $-\omega$ 也被创造出来了，即负的无穷大。

B：第 $\aleph$ 日可真够忙的。

A：接下来是下一日 —

B：你是说，第 $\aleph$ 日还不算完！

A：哦，没有完，Conway 为什么要停下来呢？我有一种预感，他才刚开了个头呢！这个过程永无止境，因为你总是可以取 $X_R$ 为空集，且取 $X_L$ 为包含所有当前已经创造出来的所有数的集合。

B：但是，在第 $\aleph$ 日以后，可以做的事似乎不多了，因为实数之间排列得都这么稠密了。宇宙的非无穷部分已经完工了，因为在两个"相邻"的实数之间，已经是没有空间再拿来放置更多的数了。

A：并非如此，Bill，我以前也是这么想的，直到你刚对这么一说我才发现不对劲。我认为，这一点正是我想和你探讨的。若取 $X_L$ 为 $\{0\}$，并取 $X_R$ 为 $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$。这个数大于 $0$，并且小于所有正实数！我们也许可以称它为 $\varepsilon$。

B：（晕）唔……还行，我能接受。但是那个有点儿太过了。我是想说，总得有个界限才是。

最让我吃惊的地方是，你的数 $\epsilon$ 事实上是在第 $\aleph$ 日被创造出来的，并非在之后。因为你也可以取 $X_R = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$。同样，像这样疯狂的数还有一大把，比如

$$\left(\{1\}, \{1\tfrac{1}{2}, 1\tfrac{1}{4}, 1\tfrac{1}{8}, 1\tfrac{1}{16}, \ldots\}\right)$$

它比 1 只大了头发丝儿那么多。

并且，我觉得在任何数的右边都有这么一个数与之紧邻，比如 $\pi$ ……不，好像不行……

A. The one just greater than $\pi$ doesn't come until the day after $\aleph$. Only terminating binary numbers get an infinitely close neighbor on $\aleph$ day.

B. On the day after $\aleph$ we're also going to get a number *between* 0 and $\epsilon$. And you think Conway was just getting started.

A. The neatest thing, Bill, is that we not only have the real numbers and infinity and all the in-betweens ... we also have rules for telling which of two numbers is larger, and for adding and subtracting them.

B. That's right. We proved all these theorems, thinking we already *knew* what we were proving—it was just a game, to derive all the old standard laws of arithmetic from Conway's few rules. But now we find that our proofs apply also to infinitely many unheard-of cases! The numbers are limited only by our imagination, and our consciousness is expanding, and ...

A. You know, all this is sort of like a religious experience for me; I'm beginning to get a better appreciation of God. Like He's everywhere ...

B. Even between the real numbers.

A. C'mon, I'm serious.

A：这个刚好大于 $\pi$ 的数，是直到第 $\aleph$ 日的下一日才会出现的。只有那些有着会终结的二进制形式的数会在第 $\aleph$ 日获得一个无限接近的相邻数。

B：在第 $\aleph$ 日的下一日，我们还会获得一个位于 0 和 $\epsilon$ 之间的数。难怪你会觉得 Conway 是刚开了个头。

A：Bill，最漂亮的地方在于，我们不仅得到了实数、无穷、以及位于它们之间的所有这些数……我们还掌握了一整套的规则，用以判定两个数中何者为大，以及对它们做加法和减法。

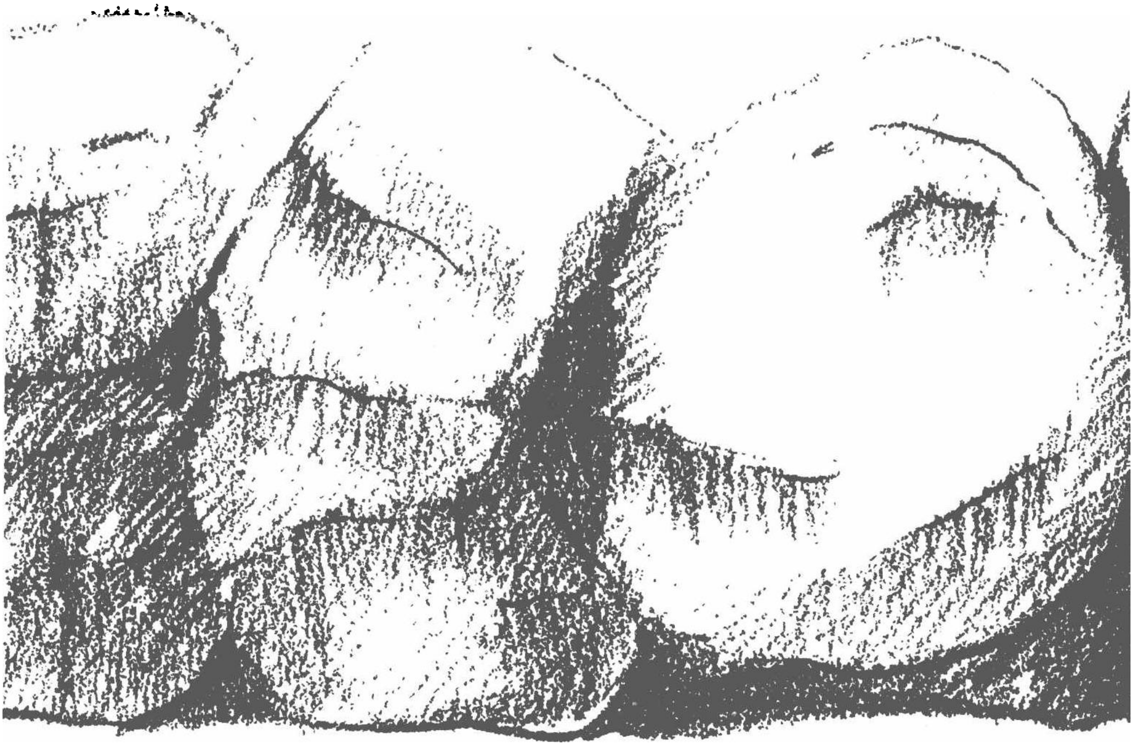
B：一点没错。我们证明了所有这些定理，这个过程中我们自认为早已了解想要证明的是什么——只是抱着游戏的心态来做这些事，却从 Conway 的寥寥数条规则中派生出全部旧有的标准算术规则来。可是现在却发现，我们的证明同样适用于无穷多种闻所未闻的情形！这些数仅仅受限于我们的想象力，我们的意识世界由此得以拓展，并且……

A：你知道吗，所有这一切对我来说简直像是宗教式的体验，我开始觉得自己更能够理解上帝了。就好像他无所不在……

B：甚至存在于实数的间隙中。

A：哎呀，人家可是认真的哟。

# 15 INFINITY



. . . . . . . .

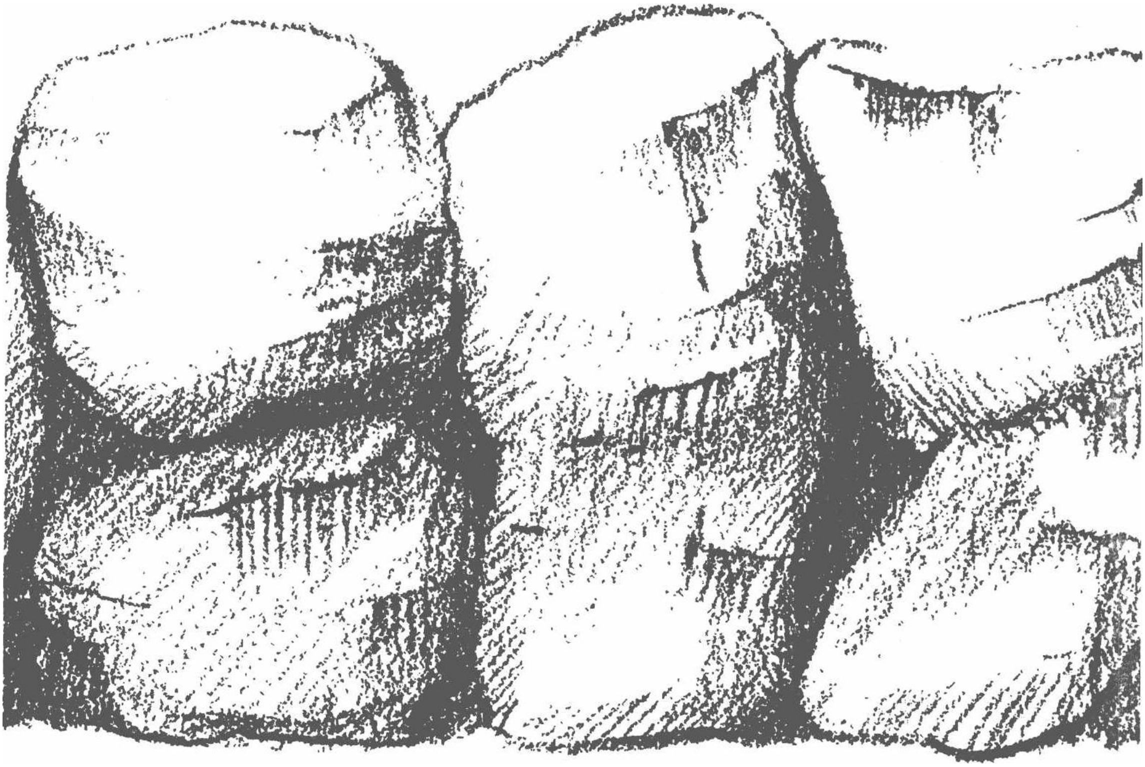
B. I've been doing a few calculations with infinity. Like, rule (3) tells us immediately that

$$\omega + 1 = (\{\omega, 2, 3, 4, 5, \dots\}, \emptyset),$$

which simplifies to

$$\omega + 1 \equiv (\{\omega\}, \emptyset).$$

# 15 玄极



…………

B：我拿无穷数做了一些运算。比如，由规则 (3) 可以立刻得到

$$\omega+1=\big(\{\omega,2,3,4,5,\ldots\},\,\emptyset\big),$$

它可以化简为

$$\omega+1\equiv\big(\{\omega\},\,\emptyset\big).$$

A. That was created on the day after ℵ day.

B. Right, and

$$\omega + 2 \equiv (\{\omega + 1\}, \emptyset)$$

on the day after. Also,

$$\omega + \tfrac{1}{2} \equiv (\{\omega\}, \{\omega + 1\}).$$

A. What about $\omega - 1$?

B. $\omega - 1$! I never thought of subtracting from infinity, because a number less than infinity is supposed to be finite. But, let's grind it out by the rules and see what happens. ... Look at that,

$$\omega - 1 \equiv (\{1, 2, 3, 4, \dots\}, \{\omega\}).$$

Of course—it's the first number created that is larger than all integers, yet less than $\omega$.

A. So *that's* what the Stone meant about an infinite number less than infinity.

Okay, I've got another one for you, what's $\omega + \pi$?

B. Easy:

$$\omega + \pi \equiv (\omega + \Pi_L, \omega + \Pi_R).$$

This was created on ... (2ℵ) day! And so were $\omega + \epsilon$ and $\omega - \epsilon$.

A. Oho! Then there must also be a number $2\omega$. I mean, $\omega + \omega$.

B. Yup,

$$\omega + \omega = (\{\omega + 1, \omega + 2, \omega + 3, \omega + 4, \dots\}, \emptyset).$$

I guess we can call this $2\omega$, even though we don't have multiplication yet, because we'll certainly prove later on that $(x + y)z \equiv xz + yz$. That means $2z \equiv (1 + 1)z \equiv z + z$.

A：这个数创造于第 $\aleph$ 日的下一日。

B：对，接下来一日创造了

$$\omega+2\equiv\big(\{\omega+1\},\emptyset\big).$$

同一天还创造了

$$\omega+\tfrac{1}{2}\equiv\big(\{\omega\},\{\omega+1\}\big).$$

A：那 $\omega-1$ 呢？

B：$\omega-1$ 呀！我从来没想过对无穷数做减法，因为一个数如果小于无穷，那就应该是有穷的。但是我们就按照规则所描述的那样机械地做，看看能得到什么结果……你瞧，

$$\omega-1\equiv\big(\{1,2,3,4,\ldots\},\{\omega\}\big)$$

当然了，这是被创造的数中首个比所有的整数都要大的数，但它却小于 $\omega$。

A：所以，那个就是岩石上说的“后出一无穷数，不及玄极”。

好，我再考你一下，$\omega+\pi$ 又是什么呢？

B：简单：

$$\omega+\pi\equiv(\omega+\Pi_L,\omega+\Pi_R).$$

这个是在第……$2\aleph$ 日被创造的！同一天还创造了 $\omega+\epsilon$ 和 $\omega-\epsilon$。

A：啊哈！这么说来，肯定还会有 $2\omega$ 这么个数。我的意思是 $\omega+\omega$。

B：肯定了，

$$\omega+\omega=\big(\{\omega+1,\omega+2,\omega+3,\omega+4,\ldots\},\emptyset\big).$$

我猜想我们应该把这个称为 $2\omega$ 吧，尽管我们还没有掌握乘法，但我们以后肯定能够证明 $(x+y)z\equiv xz+yz$。这就意味着 $2z\equiv(1+1)z\equiv z+z$。

A. Right, and

$$3\omega = \big(\{2\omega+1, 2\omega+2, 2\omega+3, 2\omega+4, \ldots\}, \emptyset\big)$$

will be created on $(3\aleph)$ day, and so on.

B. We still don't know about multiplication, but I'm willing to bet that $\omega$ times $\omega$ will turn out to be

$$\omega^2 = \big(\{\omega, 2\omega, 3\omega, 4\omega, \ldots\}, \emptyset\big).$$

A. Created on $\aleph^2$ day. Just imagine what Conway must be doing to all the smaller numbers during this time.

B. You know, Alice, this reminds me of a contest we used to have on our block when I was a kid. Every once in a while we'd start shouting about who knows the largest number. Pretty soon one of the kids found out from his dad that infinity was the largest number. But I went him one better by calling out "infinity plus one." Well, the next day we got up to infinity plus infinity, and soon it was infinity times infinity.

A. Then what happened?

B. Well, after reaching "infinityfinityfinityfinity ...," repeated as long as possible without taking a breath, we sort of gave up the contest.

A. But there are plenty more numbers left. Like

$$\omega^\omega = \big(\{\omega, \omega^2, \omega^3, \omega^4, \ldots\}, \emptyset\big).$$

And still we're only at the beginning.

B. You mean, there's $\omega^{\omega^\omega}$, $\omega^{\omega^{\omega^\omega}}$, and the limit of this, and so on. Why didn't I think of that when I was a kid?

A. It's a whole new vista. ... But I'm afraid our proofs aren't correct any more, Bill.

A：对呀，那样的话

$$3\omega = \big(\{2\omega+1, 2\omega+2, 2\omega+3, 2\omega+4, \ldots\}, \emptyset\big)$$

会在第 $3\aleph$ 日被创造出来，依此类推。

B：虽然还不知道乘法长成啥样，但是我打赌如果 $\omega$ 乘以 $\omega$ 的话，会是

$$\omega^2 = \big(\{\omega, 2\omega, 3\omega, 4\omega, \ldots\}, \emptyset\big).$$

A：它被创造于第 $\aleph^2$ 日。想像一下，Conway 在此期间对那些更小的数都做了些什么？

B：你知道吗，Alice，你这句话让我想起了我还是个小孩子的时候，在街道里举行的一场竞赛。每过一会儿，我们就大声地叫出一个数，来比比谁的最大。很快，有一个孩子就从他爸爸那里学到，无穷就是最大的数。但是，我比他更胜一筹地叫道“无穷加一”。这么一来，第二天我们就达到了无穷加无穷，很快又出现了无穷乘以无穷。

A：接下来呢？

B：接下来，直到“无穷的无穷的无穷的无穷……”就是在一口气所及的范围内重复尽可能多遍啦，后来这场竞赛也就只能不了了之了。

A：可是即使那样也还有很多数没提到呢，比如

$$\omega^\omega = \big(\{\omega, \omega^2, \omega^3, \omega^4, \ldots\}, \emptyset\big).$$

我们仍然仅仅开了个头而已。

B：你的意思是说，还有 $\omega^{\omega^\omega}$、$\omega^{\omega^{\omega^\omega}}$，以及这样以极限的方式推下去，等等。我小时候怎么就没想到这些呢？

A：这可是全新的视野……可是我觉得咱们的证明好像这么一来就不对了呢，Bill。

B. What? Not again. We already fixed them.

Oh-oh, I think I see what you're getting at. The day-sums.

A. Right. We can't argue by induction on the day-sums because they might be infinite.

B. Maybe our theorems don't even work for the infinite cases. It sure would be nice if they did, of course. I mean, what a feeling of power to be proving things about all these numbers we haven't even dreamed of yet.

A. We didn't have any apparent trouble with our trial calculations on infinite numbers. Let me think about this for awhile.

. . . . . . . .

It's okay, I think we're okay, we don't need "day-sums."

B. How do you manage it?

A. Well, remember how we first thought of induction in terms of "bad numbers." What we had to show was that if a theorem fails for $x$, say, then it also fails for some element $x_L$ in $X_L$, and then it also fails for some $x_{LL}$ in $X_{LL}$, and so on. But if every such sequence is eventually finite — I mean if eventually we must reach a case with $X_{LL\ldots L}$ empty — then the theorem can't have failed for $x$ in the first place.

B. (whistling) I see. For example, in our proof that $x + 0 = x$, we have $x + 0 = (X_L + 0, X_R + 0)$. We want to assume by induction that $x_L + 0$ has been proved equal to $x_L$ for all $x_L$ in $X_L$. If this assumption is false, then $x_{LL} + 0$ hasn't been proved equal to $x_{LL}$ for some $x_{LL}$; or, I guess, some $x_{LR}$ might be the culprit. Any counterexample would imply an infinite sequence of counterexamples.

A. All we have to do now is show that there is no infinite ancestral sequence of numbers

$$x_1, x_2, x_3, x_4, \ldots$$

B：什么？不会吧。我们已经校正了问题啦。

哦哦，我想我知道你要说的是什么了。就是创造日之和问题。

A：没错。我们不再能采用基于创造日之和的归纳法进行论证，因为这个过程可能是无穷的。

B：可能咱们的定理即使对于一些有穷的案例也行不通。如果它们过去行得通，自然是很不错的。我是说，能够证明这些之前做梦也想不到的数，这种感觉实在太给力了。

A：我们对无穷数进行试探计算时，并没有遇到明显困难。我得想想看，这是为什么。

…………

好了，我觉得其实没问题，我们根本不需要什么"创造日之和"。

B：你是怎么想的呢？

A：那个，你还记得吧，我们最开始是从"坏数"出发考虑归纳法的。我们必须得出的是若某个定理对于 $x$ 不成立，则它对于 $X_L$ 中的某 $x_L$ 也必不成立，接着对于 $X_{LL}$ 中的某 $x_{LL}$ 也必不成立，依此类推。但是若所有这种序列都是有穷的——我的意思是，如果我们必能找到一个空的 $X_{LL\ldots L}$——那么该定理从一开始就不会不成立了。

B：（吹了声口哨）我明白了。举例来说，在我们证明 $x+0=x$ 的过程中，我们先得到了 $x+0=(X_L+0, X_R+0)$。我们想通过归纳断言 $x_L+0$ 已被证明对于 $X_L$ 中的所有 $x_L$，都等于 $x_L$。因为如果这个断言不成立，则对于某些 $x_{LL}$ 就不能证明 $x_{LL}+0$ 等于 $x_{LL}$；或者，某些 $x_{LR}$ 就会成为罪魁祸首。若出现任何的反例，都会引发一个反例的无穷序列。

A：我们现在要做的一切，就是证得没有无穷祖先数列

$$x_1, x_2, x_3, x_4, \ldots$$

such that $x_{i+1}$ is in $X_{iL} \cup X_{iR}$.

B. That's a nice way to put it.

A. Also, it's true, because every number (in fact, every pseudo-number) is created out of *previously created* ones. Whenever we create a new number $x$, we can prove simultaneously that there is no infinite ancestral sequence starting with $x_1 = x$, because we have previously proved that there's no infinite sequence that proceeds from any of the possible choices of $x_2$ in $X_L$ or $X_R$.

B. That's logical, and beautiful ... But it almost sounds like you're proving the validity of induction, by using induction.

A. I suppose you're right. This must actually be an axiom of some sort, it formalizes the intuitive notion of "previously created" that we glossed over in rule (1). Yes, that's it; rule (1) will be on a rigorous footing if we formulate it in this way.

B. What you've said covers only the one-variable case. Our day-sum argument has been used for two, three, even four variables, where the induction for $(x, y, z)$ relies on things like $(y, z, x_L)$ and so on.

A. Exactly. But in every case, the induction went back to some *permutation* of the variables, with at least one of them getting an additional $L$ or $R$ subscript. Fortunately, this means that there can't be any infinite chain such as

$$(x, y, z) \to (y, z, x_L) \to (z_R, y, x_L) \to \cdots,$$

and so on; if there were, at least one of the variables would have an infinite ancestral chain all by itself, contrary to rule (1).

B. (hugging her once again) Alice, I love you, in infinitely many ways.

A. (giggling) "How do I love thee? Let me count the ways."

$1, \omega, \omega^2, \omega^\omega, \omega^{\omega^\omega}, \ldots$ .

使得 $x_{i+1}$ 属于 $X_{iL} \cup X_{iR}$。

B：听上去很美妙哇。

A：并且它是正确的呢，因为每个数（事实上，每个伪数）都是在以前已经创造的数的基础上被创造出来的。每当我们创造了一个数，则同时可以证明没有无穷祖先数列是以 $x_1 = x$ 为起始元素的，因为我们先前证明了没有无穷祖先数列可以由从 $X_L$ 或 $X_R$ 中任何可能选择的 $x_2$ 而前进下去。

B：这很符合逻辑，也很有美感……不过，好像你在使用归纳法，来证明归纳的成立。

A：我认为你说得是对的。这一条事实上必须得是某种公理才行，它把我们藏在规则 (1) 中的“以前创造的”直觉观念给形式化了。是的，就是这样。如果我们做了这样的形式化以后，规则 (1) 就立论于一个严谨的基础之上了。

B：你说的仅对于单变量的情形成立。可我们有关创造日之和的论证还被用在了双变量、三变量甚至四变量的情形呢，那里对于 $(x, y, z)$ 的归纳依赖于类似于 $(y, z, x_L)$ 这样的东西，依此类推。

A：一点儿没错。但是对于每一种情形来说，归纳法说到底都是变量的某种排列，并且其中至少一个附有下标 $L$ 或 $R$。幸运的是，这么一来，就不可能有像这样的无穷推导链

$$(x, y, z) \to (y, z, x_L) \to (z_R, y, x_L) \to \cdots,$$

如此这般。如果这样的无穷推导链存在，那么至少其中一个变量会有一个全部由它自己组成的无穷祖先数列，这与规则 (1) 矛盾。

B：（再次拥抱她）Alice，我爱你，以无穷种方式。

A：（格格地笑）“爱君何甚？待我细数。”[1]

$$1, \omega, \omega^2, \omega^\omega, \omega^{\omega^\omega}, \ldots.$$

[1] 原文“How do I love thee? Let me count the ways.”系勃朗宁夫人情诗中最脍炙人口的一句。此处取其字面意义“让我数数有多少种方式”，有双关的妙趣。

B. It still seems that we have gotten around this infinite induction in a sneaky and possibly suspicious way. Although I can't see anything wrong with your argument, I'm still leery of it.

A. As I see it, the difference is between proof and calculation. There was no essential difference in the finite case, when we were just talking about numbers created before day $\aleph$. But now there is a definite distinction between proof and the ability to calculate. There are no infinite ancestral sequences, but they can be arbitrarily long, even when they start with the same number. For example, $\omega$, $n$, $n-1$, ..., 1, 0 is a sequence of ancestors of $\omega$, for all $n$.

B. Right. I've just been thinking about the ancestral sequences of $\omega^2$. They're all finite, of course; but they can be so long, the finiteness isn't even obvious.

A. This unbounded finiteness means that we can make valid proofs, for example, that $2 \times \pi = \pi + \pi$, but we can't necessarily calculate $\pi + \pi$ in a finite number of steps. Only God can finish the calculations, but we can finish the proofs.

B. Let's see, $\pi + \pi = (\pi + \Pi_L, \pi + \Pi_R)$, which ... Okay, I see, there are infinitely many branches of the calculations but they all are at a finite distance from the starting point.

A. The neat thing about the kind of induction we've been using is that we never have to prove the "initial case" separately. The way I learned induction in school, we always had to prove P(1) first, or something like that. Somehow we've gotten around this.

B. You know, I think I understand the real meaning of induction for the first time. And I can hardly get over the fact that all of our theory is really valid, for the infinite and infinitesimal numbers as well as the finite binary ones.

A. Except possibly (T8), which talks about the "first number created" with a certain property. I suppose we could assign a num-

B：可我还是觉得这样绕过无穷归纳的方式有些见不得阳光，并且很可能包含错误。尽管我看不出你的论证有什么问题，但我仍然感觉哪里不对劲。

A：在我看来，这里面涉及的是证明和计算两者的区别。对于有穷情形，也就是在讨论第 $\aleph$ 日之前所创造的数时，这两者是没有本质的区别的。但是到了现在，证明和计算的能力之间就产生明显区别了。虽然无穷祖先数列并不存在，但它却是任意长的，即使它们以相同数作为起始元素。比如对于所有 $n$ 来说，$\omega, n, n-1, \ldots, 1, 0$ 就是关于 $\omega$ 的一个祖先数列。

B：没错，我刚才就在想 $\omega^2$ 的祖先数列是什么。它们当然都是有穷的，但是它们是如此之长，以至于它们的有穷性甚至并不那么明显。

A：这种无界的有穷性意味着，我们可以给出有效的证明，比如，$2 \times \pi = \pi + \pi$，但是我们并不一定能在有穷步内计算出 $\pi + \pi$ 来。只有上帝能完成计算，但我们却可以完成证明。

B：让我试试，$\pi + \pi = (\pi + \prod_L, \pi + \prod_R)$，这个就是……好吧，我懂了，这个计算有无穷的分支，但是所有的分支离起始点只有有穷的距离。

A：我们在用这种归纳法的妙处在于，没有必要去单独证明“初始情况”。在学校里学到的归纳法，总是要我们先证明什么 P(1) 之类的。我们总算用了一些办法绕过这个步骤了。

B：你知道吗？我觉得我今天才第一次明白了归纳法的真正含义。并且，我无法对这个事实视而不见，那就是我们这套理论真正是有效的，无论是对于无穷大还是无穷小的数，还是有着有穷二进制表示的数都成立。

A：可能 (T8) 除外，它讨论的是具有某种特性的“首个被创造的数”。我想我们可以给每一日赋以一个数，

ber to each day, like say the largest number created on that day, and order the days that way ...

B. I sort of follow you. I've noticed that a number seems to be the largest created on its day when $X_R$ is empty and $X_L$ is all the previously created numbers.

A. Maybe that explains why there was $\aleph$ day and $(\aleph + 1)$ day, but no $(\aleph - 1)$ day.

B. Yeah, I guess, but this is all too deep for me. I'm ready to tackle multiplication now, aren't you?

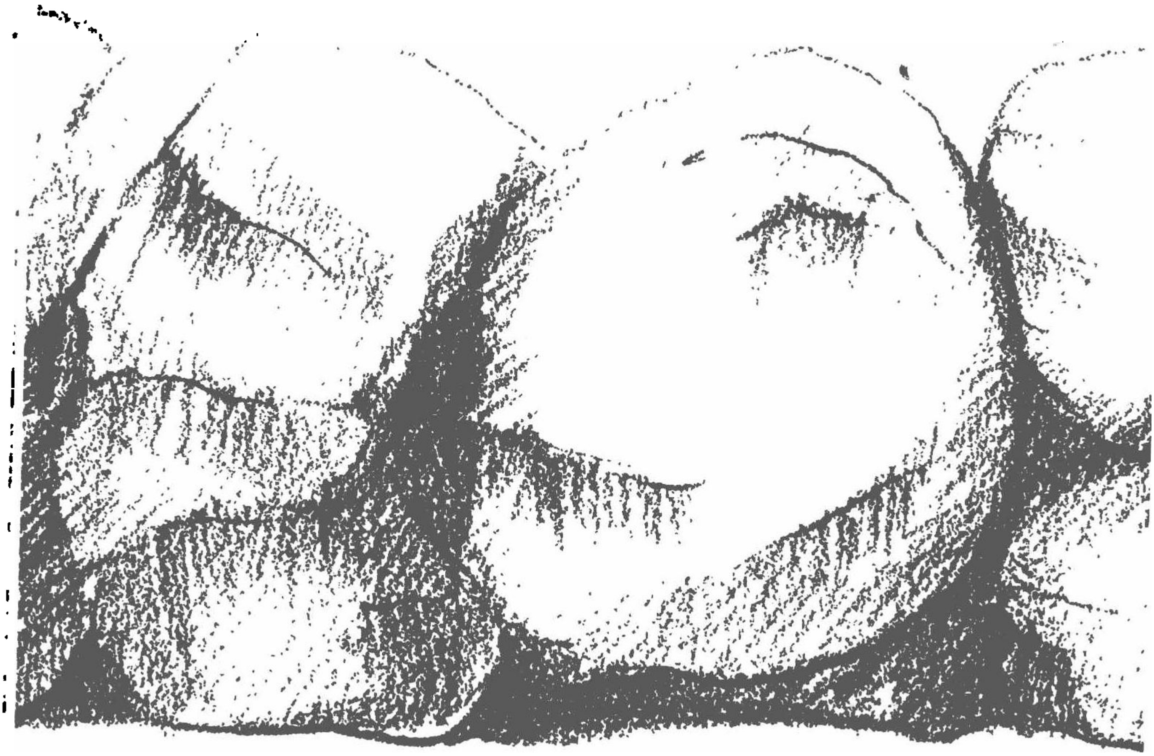
比如把当日创造的最大数赋给该日，然后就能将所有日子依此排序了……

B：我有点儿明白你的意思。我发现，似乎当 $X_R$ 为空，而 $X_L$ 为之前创造的所有数组成的集合时，该数就是当日创造的最大数。

A：也许这就解释了为什么有第 $\aleph$ 日和第 $(\aleph+1)$ 日，却没有第 $(\aleph-1)$ 日。

B：也许吧，我想，但是这些对我来说实在太过高深了。我打算着手处理乘法了，你看如何？

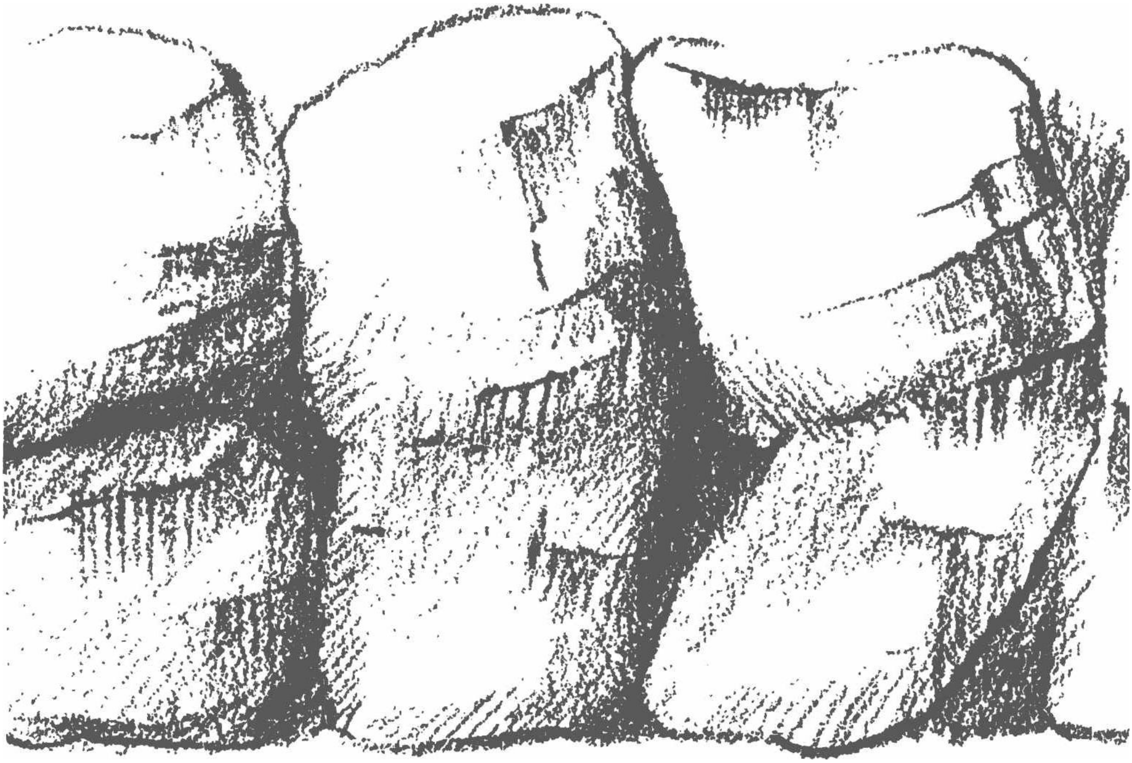
# 16 MULTIPLICATION



A. Let's see that paper where you wrote down Conway's rule for multiplication. There must be a way to put it in symbols. . . . It's complicated, but we already know what he means by "part of the same kind."

B. Alice, this is too hard. Let's try to invent our own rule for multiplication instead of deciphering that message.

# 16 乘法



A：我们来看看那张你写下了 Conway 乘法规则的纸片吧。肯定能有什么办法能把这些描述使用符号表示……真复杂呀，好在我们已经知道他说的“依其类分”是什么意思了。

B：Alice，这实在太难懂了。我们先试着来自创一些乘法规则，而不要先去解密那些文字了。

Why don't we just do like he did for addition. I mean, $xy$ should lie between $X_L y \cup xY_L$ and $X_R y \cup xY_R$. At least, it ought to do this when negative numbers are excluded.

A. But that definition would be identical to addition, so the product would turn out to be the same as the sum.

B. Whoops, so it would. ... All right, I'm ready to appreciate Conway's solution, let's look at that paper.

A. Don't feel bad about it, you've got exactly the right attitude. Remember what we said about always trying to do things by ourselves first?

B. Hah, I guess that's one lesson we've learned.

A. The best I can make out is that Conway chooses the left set of $xy$ to be all numbers of the form

$$x_L y + x y_L - x_L y_L \qquad \text{or} \qquad x_R y + x y_R - x_R y_R,$$

and the right set contains all numbers of the form

$$x_L y + x y_R - x_L y_R \qquad \text{or} \qquad x_R y + x y_L - x_R y_L.$$

You see, the left set gets the "same kinds" and the right set gets the "opposite kinds" of parts. Does this definition make any sense?

B. Lemme see, it looks weird. Well, $xy$ is supposed to be greater than its left part, so do we have

$$xy > x_L y + x y_L - x_L y_L?$$

This is like ... yeah, it's equivalent to

$$(x - x_L)(y - y_L) > 0.$$

A. That's it, the product of positive numbers must be positive! The other three conditions for $xy$ to lie between its left and

为什么我们不学学他在定义加法时采取的做法呢。我的意思是，$xy$ 应该位于 $X_Ly \cup xY_L$ 和 $X_Ry \cup xY_R$ 之间。至少，不考虑负数的话应该这样做吧。

A：但这个定义不就和加法一模一样了吗？那么这样算出来的乘积就等于加和了。

B：喔，还真是这样……那好吧，那就只能试着去理解 Conway 的方法了，我们来看看那张纸片吧。

A：别嫌不好意思呀，你所持的态度完全正确。还记得我们说过的，有关首先去自己尝试的那些话么？

B：哈哈，那个呀，是我们习得的宝贵一课。

A：我最多能够看懂到这个程度：Conway 为 $xy$ 的左集选取的是所有形如

$$x_Ly + xy_L - x_Ly_L \qquad \text{或} \qquad x_Ry + xy_R - x_Ry_R,$$

的数，而右集则选取形如

$$x_Ly + xy_R - x_Ly_R \qquad \text{或} \qquad x_Ry + xy_L - x_Ry_L.$$

的数。你看，左集选取的是“同部”而右集选取的是“异部”。这个定义是不是挺合理的？

B：我想想，这看上去有点儿怪。那个，$xy$ 按理说应该是要大于它的左集的，那末，以下不等式是否成立？

$$xy > x_Ly + xy_L - x_Ly_L$$

有点儿眼熟……对啦，这不就等价于

$$(x - x_L)(y - y_L) > 0.$$

A：这就对了，正数之积为正嘛。另外三种说明 $xy$ 位于其左

right sets are essentially saying that

$$(x_R - x)(y_R - y) > 0,$$
$$(x - x_L)(y_R - y) > 0,$$
$$(x_R - x)(y - y_L) > 0.$$

Okay, the definition looks sensible, although we haven't proved anything.

B. Before we get carried away trying to prove the main laws about multiplication, I want to check out a few simple cases just to make sure. Let's see ...

$$xy = yx; \tag{T20}$$
$$0y = 0; \tag{T21}$$
$$1y = y. \tag{T22}$$

Those were all easy.

A. Good, zero times infinity is zero. Another easy result is

$$-(xy) = (-x)y. \tag{T23}$$

B. Right on. Look, here's a fun one:

$$\tfrac{1}{2}x \equiv \big(\tfrac{1}{2}X_L \cup (x - \tfrac{1}{2}X_R),\ (x - \tfrac{1}{2}X_L) \cup \tfrac{1}{2}X_R\big). \tag{T24}$$

A. Hey, I've always wondered what *half of infinity* was.

B. Half of infinity! ... Coming right up.

$$\tfrac{1}{2}\omega \equiv \big(\{1, 2, 3, 4, \ldots\},$$
$$\{\omega - 1, \omega - 2, \omega - 3, \omega - 4, \ldots\}\big).$$

It's interesting to prove that $\frac{1}{2}\omega + \frac{1}{2}\omega \equiv \omega$. ... Wow, here's another neat result:

$$\epsilon\omega \equiv 1.$$

右两集之间的描述本质上就是说

$$(x_R - x)(y_R - y) > 0,$$
$$(x - x_L)(y_R - y) > 0,$$
$$(x_R - x)(y - y_L) > 0.$$

哇,定义看上去挺合理,尽管我们还什么都没证明呢。

B：在我们将注意力转移到乘法的主要定理证明上去之前,我想先把它在一些平凡的例子上试用一下以确认。比如……

$$xy = yx; \tag{T20}$$
$$0y = 0; \tag{T21}$$
$$1y = y. \tag{T22}$$

这些都很简单了。

A：不错,零乘以无穷大的结果是零。另一个马上就能得到的结果是

$$-(xy) = (-x)y. \tag{T23}$$

B：完全正确。你看,下面这个就有意思了

$$\tfrac{1}{2}x \equiv \left(\tfrac{1}{2}X_L \cup (x - \tfrac{1}{2}X_R), (x - \tfrac{1}{2}X_L) \cup \tfrac{1}{2}X_R\right) \tag{T24}$$

A：嘿,我一直想不明白,*无穷的一半是多少*。

B：无穷的一半!……马上就算。

$$\tfrac{1}{2}\omega \equiv \left(\{1, 2, 3, 4, \ldots\},\right.$$
$$\left.\{\omega - 1, \omega - 2, \omega - 3, \omega - 4, \ldots\}\right).$$

十分意味深长的是,我们可以证明 $\frac{1}{2}\omega + \frac{1}{2}\omega = \omega$.……喔,这里有另一个优雅的结果:

$$\epsilon\omega = 1.$$

Our infinitesimal number turns out to be the reciprocal of infinity!

A. While you were working that out, I was looking at multiplication in general. It looks a little freaky for pseudo-numbers— I found a pseudo-number $p$ for which $(\{1\}, \emptyset)p$ is not like $(\{0,1\}, \emptyset)p$, even though $(\{1\}, \emptyset)$ and $(\{0,1\}, \emptyset)$ are both like 2. In spite of this difficulty, I applied your Big Picture method and I think it is possible to prove

$$x(y+z) \equiv xy + xz, \tag{T25}$$

$$x(yz) \equiv (xy)z \tag{T26}$$

for arbitrary pseudo-numbers, and

$$\text{if} \quad x > x' \quad \text{and} \quad y > y', \quad \text{then} \quad (x - x')(y - y') > 0 \tag{T27}$$

for arbitrary numbers. It will follow that $xy$ is a number whenever $x$ and $y$ are.

B. Theorem (T27) can be used to show that

$$\text{if} \quad x \equiv y, \quad \text{then} \quad xz \equiv yz \tag{T28}$$

for all numbers. So all of these calculations we've been making are perfectly rigorous.

I guess that takes care of everything it says on the tablet. Except the vague reference to "series, and quotients, and roots."

A. Hmm ... What about division? ... I bet if $x$ is between 0 and 1, it'll be possible to prove that

$$1 - \frac{1}{1+x} \equiv \big(\{x - x^2,\ x - x^2 + x^3 - x^4,\ \ldots\},\ \{x,\ x - x^2 + x^3,\ x - x^2 + x^3 - x^4 + x^5,\ \ldots\}\big).$$

我们的无穷小数原来是无穷大数的倒数！

A：在你做出那些结果的时候，我在考虑乘法的一般规律。看上去它用于伪数时显得比较反常——我发现对于某伪数 $p$ 而言，$(\{0,1\},\emptyset)p$ 并不相似于 $(\{0,1\},\emptyset)p$，尽管 $(\{1\},\emptyset)$ 和 $(\{0,1\},\emptyset)$ 都相似于 2。但先不管这个困难所在，我运用了一下你的全景框架法以后觉得，对于任何伪数我们可能都可以证明

$$x(y+z) \equiv xy + xz, \tag{T25}$$

$$x(yz) \equiv (xy)z, \tag{T26}$$

且对于任何数都可以证明

$$\text{若} \quad x > x' \quad \text{且} \quad y > y',$$
$$\text{则} \quad (x - x')(y - y') > 0. \tag{T27}$$

从这个可以得出推论：$xy$ 一定是数，而无论 $x$ 和 $y$ 是否是数。

B：定理 (T27) 可以用来证得

$$\text{若} \quad x \equiv y, \quad \text{则} \quad xz \equiv yz \tag{16.1}$$

对于所有数成立。所以，我们进行过的一切计算在严格性方面堪称完美。

我想，现在石板上的一切都已经完成了。除了那句含糊地提到的“为级数、为商数、为方根”尚无着落。

A：唔……除法是什么样的？……我敢打赌，如果 $x$ 位于 0 和 1 之间的话，可能可以证明：

$$1 - \frac{1}{1+x} \equiv \big(\{x - x^2,\ x - x^2 + x^3 - x^4,\ \ldots\},\ \{x,\ x - x^2 + x^3,\ x - x^2 + x^3 - x^4 + x^5,\ \ldots\}\big).$$

At least, this is how we got $\frac{1}{3}$, for $x = \frac{1}{2}$. Perhaps we'll be able to show that every nonzero number has a reciprocal, using some such method.

B. Alice! Feast your eyes on this!

$$\sqrt{\omega} \equiv \left(\{1, 2, 3, 4, \ldots\}, \left\{\frac{\omega}{1}, \frac{\omega}{2}, \frac{\omega}{3}, \frac{\omega}{4}, \ldots\right\}\right);$$

$$\sqrt{\epsilon} \equiv \left(\{\epsilon, 2\epsilon, 3\epsilon, 4\epsilon, \ldots\}, \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}\right).$$

A. (falling into his arms) Bill! Every discovery leads to more, and more!

B. (glancing at the sunset) There are infinitely many things yet to do ... and only a finite amount of time ...

至少，这是我们如何通过 $x=\frac{1}{2}$ 计算出 $\frac{1}{3}$ 的过程。也许我们就用这样的办法，可以证明任何非零数都有倒数。

B：Alice！看看这个会不会让你大饱眼福！

$$\sqrt{\omega} \equiv \left(\{1,2,3,4,\ldots\},\left\{\frac{\omega}{1},\frac{\omega}{2},\frac{\omega}{3},\frac{\omega}{4},\ldots\right\}\right);$$
$$\sqrt{\epsilon} \equiv \left(\{\epsilon,2\epsilon,3\epsilon,4\epsilon,\ldots\},\left\{\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right\}\right).$$

A：（投入他的怀抱）Bill！每个发现都会导致更多的发现，然后又是更多！

B：（凝视着夕阳）需要做的工作是无穷的……而时间却是有穷的呢……

The reader may have guessed that this is not a true story. However, "J. H. W. H. Conway" does exist — he is Professor John Horton Conway of Cambridge University. The real Conway has established many remarkable results about these "extraordinal" numbers, besides what has been mentioned here. For example, every polynomial of odd degree, with arbitrary numbers as coefficients, has a root. Also, every pseudo-number $p$ corresponds to a position in a two-person game between players Left and Right, where the four relations

$$\begin{array}{ll} p > 0, & p < 0, \\ p = 0, & p \parallel 0 \end{array}$$

correspond respectively to the four conditions

| | |
|---:|:---|
| Left wins, | Right wins, |
| Second player wins, | First player wins |

starting at position $p$. [See his incredible book *On Numbers and Games* (London: Academic Press, 1976), as well as the two-volume sequel by Berlekamp, Guy, and Conway, *Winning Ways* (London: Academic Press, 1982).]

The theory is still very much in its infancy, and the reader may wish to play with some of the many unexplored topics: What can be said about logarithms? continuity? multiplicative properties of pseudo-numbers? generalized diophantine equations? etc.

读者们也许已经猜到,这里所讲述的并不是一个真实的故事。但是,"J. H. W. H. Conway"却实有其人——他就是剑桥大学的 John Horton Conway 教授。现实世界中的 Conway 教授在研究这种具有"额外数序"之数的方面得到了许多引人注目的结果,包括本书中提到的部分。比如,每个带任意系数的奇次多项式,都有方根。还有,每个伪数 $p$,都对应着在一对左右玩家之间进行的二人博弈中的一个位置,其中四种关系

$$p > 0, \qquad p < 0$$
$$p = 0, \qquad p \parallel 0$$

分别对应四种条件

左胜,　右胜,
后者胜,　前者胜

而博弈起始于位置 $p$。〔参见他的精彩著作《论数字和博弈》(*On Numbers and Games*, London: Academic Press, 1976),以及由 Berlekamp、Guy 和 Conway 合著的两卷本《取胜之道》(*Winning Ways*, London: Academic Press, 1982)。〕

整套理论仍然处在非常初步的阶段,读者可能会有兴趣研究大量未知问题中的若干个:在其中如何建立对数概念和连续性概念?伪数在乘法下会表现出哪些性质?还有广义丢番图方程[1]呢?等等。

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[1] 广义丢番图方程 (*generalized Diophantine equation*),即为形如 $\sum_{n=1} a_n x_n^n = c$,其中 $a_n, c \in Z$ 的不定方程。方程左边的表达式可能为有限项,也可能为无限项。有关这个主题,可以参阅 W. M. Schmidt, *Diophantine approximations and Diophantine equations*, Springer Verlag, 2000 等。

## POSTSCRIPT

The late Hungarian mathematician Alfréd Rényi composed three stimulating "Dialogues on Mathematics," which were published by Holden–Day of San Francisco in 1967. His first dialogue, set in ancient Greece about 440 B.C., features Socrates and gives a beautiful description of the nature of mathematics. The second, which supposedly takes place in 212 B.C., contains Archimedes's equally beautiful discussion of the applications of mathematics. Rényi's third dialogue is about mathematics and science, and we hear Galileo speaking to us from about A.D. 1600.

I have prepared *Surreal Numbers* as a mathematical dialogue of the 1970s, emphasizing the nature of creative mathematical explorations. Of course, I wrote this mostly for fun, and I hope that it will transmit some pleasure to its readers, but I must admit that I also had a serious purpose in the back of my mind. Namely, I wanted to provide some material that would help to overcome one of the most serious shortcomings in our present educational system, the lack of training for research work; there is comparatively little opportunity for students to experience how new mathematics is invented, until they reach graduate school.

I decided that creativity can't be taught using a textbook, but that an "anti-text" such as this novelette might be useful. I therefore tried to write the exact opposite of Landau's *Grundlagen der Analysis*; my aim was to show how mathematics can be "taken out of the classroom and into life," and to urge readers to try their own hands at exploring abstract mathematical ideas.

The best way to communicate the techniques of mathematical research is probably to present a detailed case study. Conway's recent approach to numbers struck me as the perfect vehicle for illustrating the important aspects of mathematical explorations, because it is a rich theory that is almost self-contained, yet with close ties to both algebra and analysis, and because it is still largely unexplored.

In other words, my primary aim is not really to teach Conway's theory, it is to teach how one might go about developing such a theory. Therefore, as the two characters in this book gradually explore and build up Conway's number system, I have recorded

# 跋

已故匈牙利数学家 Alfréd Rényi 写过三篇趣味盎然的“数学对话”,发表在 1967 年的旧金山学术期刊 Holden-Day 上。他写的第一篇对话,背景取自古希腊,时间大约在公元前 440 年,彰显了苏格拉底这个人物,并出色地描述了数学的本质。第二篇对话,据称发生在公元前 212 年,包括了阿基米德同样出色的对数学应用的描述。Rényi 的第三篇对话是有关数学和科学的,我们从中聆听到了伽利略在公元 1600 年向我们发表的高论。

我把本书设定为发生在 20 世纪 70 年代的一场数学对话,主要强调的是带有创新性的数学探索的本质。当然,我写作这篇东西主要是出于兴趣,并且我也希望它给读者诸君带来快乐,但我必须承认,我的内心深处有着更为严肃的目的。这就是,我想提供一些材料,它们会有助于克服在我们当前的教育系统中存在着的最严重的毛病,即缺乏对于研究工作的训练。相对而言,学生们在研究生阶段之前体验新的数学如何被发明出来的可能,是微乎其微的。

我认为,创新是无法通过教科书习得的,但是一本“反教科书”类的小说体图书却可能奏效。因此,我尝试着采用了一种与 Landau 的《微积分基础》(*Grundlagen der Analysis*) 完全背道而驰的方式来写作。我的目标是说明数学家如何能够“走出教室,融入生活”,并且鼓励读者亲手尝试去探索抽象的数学思想。

传授数学研究技术的最佳方法,可能就是给出一个详细的案例研究。Conway 教授最近有关数的思路触发了我的灵感,让我觉得这就是演示数学探索的完美工具,因为这是一种近乎自成体系的丰富理论的同时,又和代数与分析有着紧密的联系,还因为它尚有一大部分是未经探索的。

换言之,我的主要目标并不是真的要教给读者 Conway 教授的理论,而是想让读者学到,一个人要如何着手来研究出这么一套理论来。因此,当本书中的两个主角一点点地进行探索,并建立起 Conway 的数系时,我会记下

their false starts and frustrations as well as their good ideas and triumphs. I wanted to give a reasonably faithful portrayal of the important principles, techniques, joys, passions, and philosophy of mathematics, so I wrote the story as I was actually doing the research myself (using no outside sources except a vague memory of a lunchtime conversation I had had with John Conway almost a year earlier).

I have intended this book primarily for college mathematics students at about the sophomore or junior level. Within a traditional math curriculum it can probably be used best either (a) as supplementary reading material for an "Introduction to Abstract Mathematics" course or a "Mathematical Logic" course; or (b) as the principal text in an undergraduate seminar intended to develop the students' abilities for doing independent work.

Books that are used in classrooms are usually enhanced by exercises. So at the risk of destroying the purity of this "novel" approach, I have compiled a few suggestions for supplementary problems. When used with seminars, such exercises should preferably be brought up early in each class hour, for spontaneous class discussions, instead of being assigned as homework.

1. After Chapter 3. What is "abstraction," and what is "generalization"?

2. After Chapter 5. Assume that $g$ is a function from numbers to numbers such that $x \leq y$ implies $g(x) \leq g(y)$. Define

$$f(x) = \big(f(X_L) \cup \{g(x)\},\ f(X_R)\big).$$

Prove that $f(x) \leq f(y)$ if and only if $x \leq y$. Then in the special case that $g(x)$ is identically 0, evaluate $f(x)$ for as many numbers as you can. [*Note:* After Chapter 12, this exercise makes sense also when "numbers" are replaced by "pseudo-numbers."]

3. After Chapter 5. Let $x$ and $y$ be numbers whose left and right parts are "like" but not identical. Formally, let

$$f_L\colon X_L \to Y_L, \qquad f_R\colon X_R \to Y_R,$$
$$g_L\colon Y_L \to X_L, \qquad g_R\colon Y_R \to X_R$$

他们失败的起点以及情绪的焦虑，但也会记下他们有价值的思想以及胜利的喜悦。我想给出一张在相当程度上可信的图景，以展示数学的重要原理、技术、乐趣、热情和哲学，所以我是在一边写着这个故事的同时，一边自己在做着研究（完全没有参考外部文献，仅凭着对于约一年前我和 John Conway 共进午餐时所进行的一番对话所留下的模糊印象）。

本书的初衷主要是为了数学专业的本科生而写，大致对应的年级是大二或大三。如果按照美国传统基础课程的说法，本书既可用作 (a)"近世代数导论"课程或"数理逻辑"课程的补充读物，也可用作 (b) 本科阶段意在培养学生独立工作能力的研讨班所使用的主要教科书。

凡用作教学的书，通常都会以练习来增强其功效。所以，冒着破坏这种"小说手法"的纯粹性之大不韪，我汇集了一些供大家参考的补充习题。如果将其用于研讨班，则最好能够在每课时前提出，用于课堂讨论，而不要当成作业布置下去。

1. 第 3 章读后。什么是"抽象"，什么是"概括"？

2. 第 5 章读后。设 $g$ 为一函数，会将作为自变量之数映射至所有满足 $x \le y$ 蕴涵 $g(x) \le g(y)$ 的数。定义

$$f(x) = \big(f(X_L) \cup \{g(x)\}, f(X_R)\big)$$

证明 $f(x) \le f(y)$ 当且仅当 $x \le y$。在 $g(x)$ 恒等于 0 特殊情形下，对尽可能多的数计算 $f(x)$ 的值。〔附记：第 12 章读后，本练习在把"数"换成"伪数"时仍然有意义。〕

3. 第 5 章读后。令 $x$ 和 $y$ 为数，且其左部和右部"相似"但并不相同。形式上，令

$$f_L : X_L \to Y_L, \qquad f_R : X_R \to Y_R,$$
$$g_L : Y_L \to X_L, \qquad g_R : Y_R \to X_R$$

be functions such that $f_L(x_L) \equiv x_L$, $f_R(x_R) \equiv x_R$, $g_L(y_L) \equiv y_L$, $g_R(y_R) \equiv y_R$. Prove that $x \equiv y$. [Alice and Bill did not realize that this lemma was important in some of their investigations; they assumed it without proof. The lemma holds also for pseudo-numbers.]

4. After Chapter 6. When we are developing the theory of Conway's numbers from these few axioms, is it legitimate to use the properties we already "know" about numbers in the proofs? (For example, is it okay to use subscripts like $i-1$ and $j+1$?)
5. After Chapter 9. Find a complete formal proof of the general pattern after $n$ days. [This problem makes an interesting exercise in the design of notations. There are many possibilities, and the students should strive to find a notation that makes a rigorous proof most understandable, in the sense that it matches Alice and Bill's intuitive informal proof.]
6. After Chapter 9. Is there a simple formula telling the day on which a given binary number is created?
7. After Chapter 10. Prove that $x \equiv y$ implies $-x \equiv -y$.
8. After Chapter 12. Establish the value of $x \oplus y$ for as many $x$ and $y$ as you can.
9. After Chapter 12. Change rules (1) and (2), replacing $\not\geq$ by $<$ in all three places; and add a new rule:

$$x < y \qquad \text{if and only if} \qquad x \leq y \qquad \text{and} \qquad y \not\leq x.$$

Now develop the theory of Conway's numbers from scratch, using these definitions. [This question leads to a good review of the material in the first three chapters; the arguments have to be changed in several places. The major hurdle is to prove that $x \leq x$ for all numbers; there is a fairly short proof, not easy to discover, which I prefer not to reveal here. The students should be encouraged to discover that the new $<$ relation is not identical to Conway's, with respect to pseudo-numbers (although of course it is the same for all numbers). In the new situation, the law $x \leq x$ does not always hold; and if

$$x = \big(\{(\{0\},\{0\})\},\emptyset\big),$$

为函数，使得 $f_L(x_L) \equiv x_L$、$f_R(x_R) \equiv x_R$、$g_L(y_L) \equiv y_L$、$g_R(y_R) \equiv y_R$。证明 $x \equiv y$。〔Alice 和 Bill 并未意识到这条引理在他们某些想法中的重要性，而是未加证明地断言了它的正确性。本引理对于伪数同样成立。〕

4. 第 6 章读后。在我们从这几条公理出发来研发 Conway 数论时，在证明中运用我们已经“知道”的数的性质，合法吗？（比如，我们可以使用像 $i-1$ 和 $j+1$ 这样的下标吗？）

5. 第 9 章读后。为过了 $n$ 日后创造出来的数的模式寻找完全形式化的证明。〔这是个练习记法设计的有趣问题。它的答案多种多样，学生们应该竭尽全力去寻找合适的记号以使得严格的证明变得最易理解，在某种意义上应与 Alice 和 Bill 从直觉出发的非形式证明相吻合。〕

6. 第 9 章读后。能否找到一个简单的公式，算出任意给定的二进制数是在哪一日被创造出来的？

7. 第 10 章读后。证明 $x \equiv y$ 蕴涵 $-x \equiv -y$。

8. 第 12 章读后。对尽可能多的 $x$ 和 $y$ 求 $x \oplus y$ 的值。

9. 第 12 章读后。改变规则 (1) 和 (2)，将“$\not\geq$”由“$<$”在所有三处替换，并增加一条新规则：

$$x < y, \qquad \text{当且仅当} \qquad x \leq y \qquad \text{且} \qquad y \not\leq x.$$

然后使用这些定义，将 Conway 数论从头再建立一遍。〔这个问题会引发对于第三章材料的一次仔细检讨。论证过程要在若干处加以修订。主要的障碍在于证明对于所有的数，$x \leq x$ 都成立。有一个相当短小的证明，却不那么容易找到，我也不想在这里公布答案。要鼓励学生们去发现，这个新的 $<$ 关系和 Conway 的那个对于伪数来说并不完全等同（尽管对于所有的数来说是完全一样的）。在新的前提下，$x \leq x$ 这条定律并不总是成立。并且若

$$x = (\{(\{0\}, \{0\})\}, \emptyset),$$

we have $x \equiv 0$ in Conway's system but $x \equiv 1$ in the new one! Conway's definition has nicer properties but the new relation is instructive.]

10. After Chapter 13. Show how to avoid Alice and Bill's circularity problem another way, by eliminating $\text{III}(z, Z_L, y)$ and $\text{III}(Z_R, z, x)$ from the requirements needed to prove $\text{II}(x, y, z)$. In other words, prove directly that we can't have $z + y \le z_L + y$ for any $z_L$.
11. After Chapter 14. Determine the "immediate neighborhood" of each real number during the first few days after ℵ day.
12. After Chapter 15. Construct the largest infinite numbers you can think of, and also the smallest positive infinitesimals.
13. After Chapter 15. Does it suffice to restrict $X_L$ and $X_R$ to countable sets? [This question is difficult but it may lead to an interesting discussion. Instructors can prepare themselves by boning up on ordinal numbers.]
14. Almost anywhere. What are the properties of the operation defined by

$$x \circ y = (X_L \cap Y_L,\ X_R \cap Y_R)?$$

[The class should discover that this is *not* $\min(x, y)$.] Many other operations are interesting to explore, for example when $x \circ y$ is defined to be

$$(X_L \circ Y_L, X_R \cup Y_R)$$
$$\text{or} \qquad \big((X_L \circ y) \cup (x \circ Y_L),\ X_R \cup Y_R\big).$$

15. After Chapter 16. If $X$ is the set of *all* numbers, show that $(X, \emptyset)$ is not equivalent to *any* number. [There are paradoxes in set theory unless care is taken. Strictly speaking, the class of all numbers isn't a set. Compare this problem with the "set of all sets" paradoxes.]
16. After Chapter 16. Call $x$ a *generalized integer* if

$$x \equiv \big(\{x - 1\}, \{x + 1\}\big).$$

那么在 Conway 的数系中有 $x \equiv 0$,而在新的数系中却成了 $x \equiv 1$! Conway 的定义有着更好的性质,但是新的关系却是很有启发性的。]

10. 第 13 章读后。用另一种方法说明怎样规避 Alice 和 Bill 所遭遇的循环论证,通过从证明 $(x, y, z)$ 的前提条件中去除 $(z, Z_L, y)$ 和 $(Z_R, z, x)$。换言之,直接证明对于任何 $z_L$, $z + y \le z_L + y$ 都不成立。
11. 第 14 章读后。推算出在第 ℵ 日后的数日,每个实数的"最近邻数"是多少。
12. 第 15 章读后。构造你能够想到的最大无穷数,以及最小的正无穷小数。
13. 第 15 章读后。将 $X_L$ 和 $X_R$ 限制为可数集合,是否已经完备?〔本问题很有难度,但是可能会引发有意思的讨论。讲师可通过补习序数知识来做课前准备。〕
14. 几乎在任何章节处。下式定义的运算具有何种性质?

$$x \circ y = (X_L \cap Y_L, X_R \cap Y_R)$$

〔研讨班应该会发现这个运算并非 $\min(x, y)$。〕还有很多其他运算探索起来令人兴致大增,比如,当 $x \circ y$ 采取如下定义时。

$$(X_L \circ Y_L, X_R \cup Y_R)$$

$$\text{或} \quad \big((X_L \circ y) \cup (x \circ Y_L),\ X_R \cup Y_R\big).$$

15. 第 16 章读后。若 $X$ 是所有数构成的集合,说明 $(X, \emptyset)$ 不等价于任何数。〔若不加小心,集合论中就会出现悖论。严格地说,全体数构成的等价类并非集合。将本问题与"全体集合的集合"悖论作一对比。〕
16. 第 16 章读后。称 $x$ 为一广义整数,若

$$x \equiv \big(\{x-1\}, \{x+1\}\big).$$

Show that generalized integers are closed under addition, subtraction, and multiplication. They include the usual integers $n$, as well as numbers like $\omega \pm n$, $\frac{1}{2}\omega$, etc. [This exercise is due to Simon Norton.]

17. After Chapter 16. Call $x$ a *real number* if $-n < x < n$ for some (nongeneralized) integer $n$, and if

$$x \equiv \bigl(\{x-1, x-\tfrac{1}{2}, x-\tfrac{1}{4}, \ldots\},\ \{x+1, x+\tfrac{1}{2}, x+\tfrac{1}{4}, \ldots\}\bigr).$$

Prove that the real numbers are closed under addition, subtraction, and multiplication, and that they are isomorphic to real numbers defined in more traditional ways. [This exercise and those that follow were suggested by John Conway.]

18. After Chapter 16. Change rule (1), allowing $(X_L, X_R)$ to be a number only when $X_L \not\geq X_R$ and the following condition is satisfied:

$X_L$ has a greatest element or is null if and only if $X_R$ has a least element or is null.

Show that precisely the real numbers (no more, no less) are created in these circumstances.

19. After Chapter 16. Find a pseudo-number $p$ such that $p + p \equiv \bigl(\{0\}, \{0\}\bigr)$. [This question is surprisingly difficult, and it leads to interesting subproblems.]

20. After Chapter 15 or 16. The pseudo-number $\bigl(\{0\}, \{(\{0\}, \{0\})\}\bigr)$ is $> 0$ and $< x$ for all positive numbers $x$. Thus it is *really* infinitesimal! But $\bigl(\{0\}, \{(\{0\}, \{-1\})\}\bigr)$ is smaller yet. And any pseudo-number $p > 0$ is greater than $\bigl(\{0\}, \{(\{0\}, \{-x\})\}\bigr)$ for some suitably large number $x$.

21. After Chapter 16. For any number $x$ define

$$\omega^x = \Bigl(\{0\} \cup \{n\omega^{x_L} \mid x_L \in X_L,\ n = 1, 2, 3, \ldots\},$$
$$\Bigl\{\frac{1}{2^n}\omega^{x_R} \Bigm| x_R \in X_R,\ n = 1, 2, 3, \ldots\Bigr\}\Bigr).$$

Prove that $\omega^x\omega^y = \omega^{x+y}$.

说明广义整数在加法、减法和乘法下封闭。包括通常意义下的整数 $n$,还有像 $\omega \pm n$ 和 $\frac{1}{2}\omega$ 这样的数,等等。〔本习题由 Simon Norton 提供。〕

17. 第 16 章读后。当对某些(非广义)整数 $n$,取 $x$ 使得 $-n < x < n$,且当

$$x \equiv \left(\{x-1, x-\tfrac{1}{2}, x-\tfrac{1}{4}, \ldots\}, \{x+1, x+\tfrac{1}{2}, x+\tfrac{1}{4}, \ldots\}\right)$$

时,证明 $x$ 为一实数,并证明实数在加法、减法和乘法下封闭,并且它们和采用更传统的方式定义的实数同构。〔本习题,以及下面的所有习题,由 John Conway 提供。〕

18. 第 16 章读后。改变规则 (1),仅在 $X_L \not\geq X_R$ 且满足以下条件时允许 $(X_L, X_R)$ 成为数:

$X_L$ 中有一最大数或为空,当且仅当

$X_R$ 中有一最小数或为空

说明在这些条件下,创造出来的恰为实数(既不多也不少)。

19. 第 16 章读后。求一伪数 $p$,使得 $p+p \equiv \left(\{0\}, \{0\}\right)$。〔本习题出奇地困难,并且它会带出有意思的子问题。〕

20. 第 15 或 16 章读后。伪数 $(\{0\}, \{(\{0\}, \{0\})\})$ 是 $> 0$ 并且对所有正数 $x$ 都有 $< x$ 的。综上,它的确是无穷小的。但是 $\left(\{0\}, \{(\{0\}, \{-1\})\}\right)$ 还要更小些。并且,总有充分大的 $x$ 使得任意伪数 $p > 0$ 大于 $\left(\{0\}, \{(\{0\}, \{-x\})\}\right)$。

21. 第 16 章读后。对任意数 $x$,定义

$$\omega^x = \left(\{0\} \cup \{n\omega^{x_L} \mid x_L \in X_L,\, n = 1, 2, 3, \ldots\},\right.$$
$$\left.\left\{\frac{1}{2^n}\omega^{x_R} \,\middle|\, x_R \in X_R,\, n = 1, 2, 3, \ldots\right\}\right).$$

证明 $\omega^x \omega^y = \omega^{x+y}$。

22. After Chapter 16. Explore the properties of the symmetric pseudo-numbers $S$ such that

$$(P_L, P_R) \in S \qquad \text{if and only if} \qquad P_L = P_R \subseteq S.$$

In other words, the elements of $S$ have identical left and right sets, and so do the elements of their left and right parts, proceeding recursively. Show that $S$ is closed under addition, subtraction, and multiplication. Explore further properties of $S$; for example, how many unlike elements of $S$ are created on each day, and is their arithmetic interesting in any way? [This open-ended problem is perhaps the best on the entire list, because an extremely rich theory is lurking here.]

I will send hints to the solutions of exercises 9, 19, and 22 to any bona fide teachers who request them by writing to me at Stanford University.

Now I would like to close this postscript with some suggestions addressed specifically to teachers who will be leading a seminar based on this book. (All other people should please stop reading, and close the book at once.)

Dear Teacher: Many topics for class discussion are implicit in the story. The first few chapters will not take much time, but before long you may well be covering less than one chapter per class hour. It may be a good idea for everyone to skim the whole book very quickly at first, because the developments at the end are what really make the beginning interesting. One thing to stress continually is to ask the students to "distill off" the important general principles, the *modus operandi*, of the characters. Why do they approach the problem as they do, and what is good or bad about their approaches? How does Alice's "wisdom" differ from Bill's? (Their personalities are distinctly different.) Another ground rule for the students is that they should check over the mathematical details that often are only hinted at; this is the only way that a reader can really learn what is going on in the book. Students should preferably tackle problems first by themselves before reading on. The appearance of an ellipsis "..." often means that the characters were thinking (or writing), and the reader should do the same.

22. 第 16 章读后。探索对称伪数 $S$ 的性质，其中 $S$ 满足

$$(P_L, P_R) \in S, \qquad \text{当且仅当} \qquad P_L = P_R \subseteq S.$$

换言之，$S$ 中的元素有着一模一样的左集和右集，而其左部和右部中的元素也有这样的性质，依此递归下去。试说明 $S$ 在加法、减法和乘法下封闭。探索 $S$ 的更多性质，比如，每日有多少 $S$ 中彼此不相似的元素被创造出来，它们的四则运算有否任何意义上的有趣之处？〔这道开放式的习题可能是全部习题中最好的一道，因为一套内容极其丰富的理论就藏身于此。〕

任何真正的教师写信到斯坦福大学向我索要的话，我都将寄送习题 9、19 和 22 的解法提示。[1]

写到这里，我想以针对那些采用本书来执教研讨班的教师的一些建议来结束。（其他读者看到这里，就可以合上书卷了。）

敬爱的教师们：这个故事里隐含了大量可供班级研讨的课题。开头的若干章不会占用许多时间，但是不用过多久，可能你在一个课时内讨论一章的内容就显得远远不够。快速地泛读全书可能是个不错的主意，因为在末尾部分的研究才是使得前面部分的内容显得妙趣横生的原因所在。需要突出的一点是要让学生们“提炼”出书中的人物所总结的重要普适原理和*解决问题的套路* (modus operandi)。为何他们会采用那样的方式来着手解决这个问题，这些方式好在哪里、不好在哪里？Alice 的“智慧”和 Bill 的有何不同？（他们的性格大相径庭。）另一条学生必须遵循的基本准则是他们需要仔细体会那些往往只是一带而过的数学细节，这才是读者真正能从本书中有所收获的唯一途径。学生们最好先自己将这些问题考虑考虑，然后再往下读。书中出现的一行省略号“……”往往说明书中的人物们在思考（或在纸上演算），那么读者也应该依样行事。

[1] 作者已经在自己的网站上将这些解法提示公布出来，参见 http://www-cs-faculty.stanford.edu/~knuth/sn.html，并译附后。

When holding class discussions of exercises such as those mentioned here, I have found it a good policy to limit the number of times each person is allowed to speak up. This rule keeps the loquacious people from hogging the floor and ruining the discussion; everybody gets to participate.

Another recommendation is that the course end with a three- or four-week assignment, to write a term paper that explores some topic not explicitly worked out in this book. For example, several possible topics are indicated in the open-ended exercises listed above. Perhaps the students can do their research in groups of two. The students should also be told that they will be graded on their English expository style as well as on the mathematical content; say 50–50. They must be told that a math term paper should *not* read like a typical homework paper. The latter is generally a collection of facts in tabular form, without motivation, etc., and the grader is supposed to recognize it as a proof; the former is in prose style like in math textbooks. Students can also gain experience in writing by taking turns preparing resumés of what transpires in class; then all the other students will be able to have a record of the discussions without being distracted by taking notes themselves.

In my opinion the two greatest weaknesses in our present mathematics education are the lack of training in creative thinking and the lack of practice in technical writing. I hope that the use of this little book can help make up for both of these deficiencies.

*Stanford, California*
*May 1974*

D.E.K.

*Detection is, or ought to be, an exact science,*
*and should be treated in the same cold and unemotional manner.*
*You have attempted to tinge it with romanticism,*
*which produces much the same effect as if you worked*
*a love-story or an elopement into the fifth proposition of Euclid.*

— SHERLOCK HOLMES, in *The Sign of The Four* (1888)

在针对像上面的这样习题举行班级讨论时，我发现一个有效的策略就是为每个人的高声发言限制一定的时间。这个规则可以避免那些多话的人占用过多的时间或者使讨论偏题，每个人都要参与才可以。

另外一个建议是让本课程以一个三到四周的作业来结束，任务是完成一份学期论文，来探索一些未在书中研究明白的主题。比如，其中有一些主题就在上面列举的开放式习题中提出了。也可以让学生分成两人一组来做研究。可以告诉学生，会依据他们论文的语言风格和数学内容来打分，比方说两者的权重对半开。必须告诉学生的是，一份数学学期论文不应该看上去像一份普通的家庭作业论文那样。后者基本上是一堆事实以表格形式的堆砌，毫无生气，而评分人则应该以数学证明的角度来看待这些内容；前者则是散文形式，如同数学教科书上的那样。学生也可以通过轮流写作备班级讨论之用的摘要来获取这方面的经验，这么一来，其他学生就能够在获得一份讨论记录的同时，又不会分散他们自己做笔记的精力。

在我看来，我们目前在数学教育上存在的两个最大弱点是创新思维训练的缺乏和技术写作实践的缺乏。希望这本小小的图书能够同时在这两个方面有所补阙。

*加利福尼亚州斯坦福市* D.E.K.

*1974 年 5 月*

**侦探就是，或应该是，一门精确的科学，**
**它应该以同样冷酷的、不带感情色彩的态度来对待。**
**你却试图使它带上几分的罗曼蒂克，**
**这效果就如同，**
**你将一篇恋爱小说或一场私奔掺进了欧几里得第五公设。**
**——福尔摩斯探案全集，《四个签名》(1888)**

## The Hints Promised in the Postscript to *Surreal Numbers*

9. To prove that $x \le x$ for all numbers $x$ in the new system, first prove the auxiliary lemma

$$\text{if} \qquad x < Y_R \qquad \text{and} \qquad y \not\le x \qquad \text{then} \qquad x < y,$$

by induction on $d(x)$, for all fixed $y$.

19. Perhaps the simplest answer is $p = \big(\{(\{1\},\{1\})\},\{-1\}\big)$. Conway has proved in fact that we can solve the equation $p + p = x$ for any given $x$ by taking $p = \big(\{q\},\{x-q\}\big)$ where

$$q = \big(X_{LL} \cup (x + x - X_{RR}) \cup (x + X_L - X_R), \emptyset\big).$$

22. Considering just the members of $S$ created before $\aleph$ day, the unlike numbers are precisely $g_0$, $g_1$, $g_2$, $g_3$, ..., where

$$g_n = \big(\{g_0,\ldots,g_{n-1}\},\ \{g_0,\ldots,g_{n-1}\}\big)\,.$$

Moreover, the elements $\{g_0,\ldots,g_{n-1}\}$ form a finite field (under Conway's addition and multiplication operations), for $n = 2$, 4, 16, 256, 65536, ..., $2^{2^k}$, ... !

## 在《研究之美》跋中承诺给出的解法提示

9. 欲证明对新数系中的所有数 $x$,有 $x \leq x$ 成立,则须首先证明辅助引理

$$\text{若} \quad x < Y_R \quad \text{且} \quad y \not\leq x \quad \text{则} \quad x < y,$$

方法是对于所有确定的 $y$,对 $d(x)$ 进行归纳。

19. 也许最简单的答案是 $p = \big(\{(\{1\},\{1\})\},\{-1\}\big)$。Conway 业已证明,事实上我们可以对任意的 $x$,取 $p = \big(\{q\},\{x-q\}\big)$ 作为方程 $p+p=x$ 的解,其中

$$q = \big(X_{LL} \cup (x + x - X_{RR}) \cup (x + X_L - X_R), \emptyset\big).$$

22. 仅考虑 $S$ 在第 $\aleph$ 日之前创造的成员,不符合要求的数只能是 $g_0$, $g_1$, $g_2$, $g_3$, ...,其中

$$g_n = \big(\{g_0, \ldots, g_{n-1}\}, \{g_0, \ldots, g_{n-1}\}\big).$$

更进一步,当 $n = 2, 4, 16, 256, 65536, ..., 2^{2^k}, ...$ 时,元素 $\{g_0, \ldots, g_{n-1}\}$ (在 Conway 的加法和乘法运算下)构成了一个有限域!

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